other hand, we can calculate to first order the formal change of the integral near the geodesics; it will not necessarily be formally zero, because (6.175) refers only to competing trajectories which have common endpoints in time *and* space. We find, formally,

$$(6.176) \qquad \delta \int \mathcal{L}(x^i,\dot{x}^i,\dot{t})\, dq = \int \left[\frac{\partial \mathcal{L}}{\partial x^i}\,\delta x^i + \frac{\partial \mathcal{L}}{\partial \dot{x}^i}\,\delta \dot{x}^i + \frac{\partial \mathcal{L}}{\partial \dot{t}}\,\delta \dot{t} \right] dq$$

and by integration by parts

$$(6.177) \qquad \delta \int \mathcal{L}\, dq = \left[\frac{\partial \mathcal{L}}{\partial \dot{t}}\,\delta t \right]_{t_1,P_1}^{t_2,P_2} + \int \left\{ \left[\frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dq}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i} \right) \right] \delta x^i - \frac{d}{dq}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}} \right) \delta t \right\} dq$$

since by assumption the δx^i vanish at the endpoints. Observe that the integrand vanishes by the Euler-Lagrange equations which characterize the null geodesic. We refer specially to the last term, for which

$$(6.178) \qquad \frac{d}{dq}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}} \right) = 0 \qquad \frac{\partial \mathcal{L}}{\partial \dot{t}} = l = \text{const}$$

This is indeed the Euler-Lagrange equation for $t(q)$ since $\partial\mathcal{L}/\partial t \equiv 0$. Thus (6.177) reduces to

$$(6.179) \qquad \delta \int \mathcal{L}\, dq = l[\delta t]_{t_1}^{t_2} = l\,\delta T$$

where δT is the change in travel time between P_1 and P_2 on the competing path, which allows the correct velocity of light as determined by $\mathcal{L} = 0$.

Thus, comparing our two ways of computing δJ, we arrive at the result: *The actual path of light in three-space between two given endpoints makes the travel time of light with the prescribed local velocity a stationary value among all admissible paths.* This is the well-known Fermat principle of optics, which we now see is a consequence of our null-geodesic principle.

6.8 The Schwarzschild Radius, Kruskal Coordinates, and the Black Hole

We have noted that in the Schwarzschild line element (6.53) a singularity occurs at $r = 2m$, the Schwarzschild radius; at this radius g_{11} is infinite while g_{00} is zero. Because g_{00} is zero, the spherical surface at $r = 2m$ is an infinite red shift surface, as is clear from our discussion of the red shift in Secs. 4.2 and 4.4. That is, since light emitted by a radiating atom situated on this surface would be red-shifted to zero frequency as it traveled to larger radii, the atom could not be observed.

When r becomes less than $2m$, the signs of the metric components g_{00} and g_{11} change, g_{11} becoming positive and g_{00} becoming negative. This forces us to reconsider the physical meaning of t and r as time and radial markers inside the Schwarzschild radius. Indeed a world-line along the t axis (r, θ, φ constant) has $ds^2 < 0$ and is a *spacelike* curve, while a world-line along the r axis has $ds^2 > 0$ and is a *timelike* curve. It would thus appear natural to reinterpret r as a time marker and t as a radial marker for events which occur inside the Schwarzschild radius. Since we interpret ds/c to represent the proper time along the world line of a particle, as in Sec. 4.2, we see that ds^2 must be positive along such a path. Thus a massive particle could not remain at a constant value of r inside the Schwarzschild radius since that would imply that $ds^2 < 0$ along its world-line.

These features show that $r = 2m$ is an unusual radius, but it does not follow that the intrinsic space-time geometry becomes singular at $r = 2m$. Indeed the "singularity" is associated with the choice of coordinates. This is indicated by the fact that the *invariants* $R^{\mu}{}_{\mu}$ and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ remain finite at the Schwarzschild radius, as does the determinant of the metric tensor. The Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$ is also finite, in particular the terms $R^{i}{}_{0j0}$, which correspond to relative Newtonian forces in (5.147) (see Exercises 6.2 to 6.4). We shall investigate the mathematical and physical nature of the Schwarzschild singularity in this section by studying the behavior of a falling test body and by introducing coordinates (Kruskal, 1960) in which no singularity occurs in the metric. Specifically we shall show that the Schwarzschild coordinates r and t are well suited for describing the Schwarzschild geometry in the region of greatest physical interest, $2m < r < \infty$ and $-\infty < t < \infty$, but that an alternative choice can shed light on the nature of the surface and interior of the Schwarzschild sphere at $r = 2m$.

Consider the simple case of inward radial motion in a Schwarzschild geometry. The Euler-Lagrange equations of motion for r and t as functions of the arc length s have been obtained in Sec. 6.3. Equations (6.81) and (6.82) specialized to radial motion, $h = 0$, are

$$(6.180) \qquad \dot{t} = \frac{1}{c}\left(1 - \frac{2m}{r}\right)^{-1} \qquad \dot{r}^2 = \frac{2m}{r}$$

We have chosen initial conditions so $l = 1/c$ in (6.81), so that $t = s/c$ at large r, as in special relativity. (See Sec. 7.9 for more on this.) Two descriptions of the radial motion are of interest; one is the radial position as a function of the proper time of the test body s/c, and the other is the radial position as a function of the coordinate time t. Since the proper time is that measured by an observer falling with the body while the coordinate time corresponds to the time measured by an observer at rest a large distance from the central mass, the physical significance of these two descriptions is apparent.

Let us solve first for r as a function of s. From (6.180) we obtain immediately

$$\frac{2}{3\sqrt{2m}}\left(r^{3/2} - r_0^{3/2}\right) = s_0 - s \tag{6.181}$$

where r_0 is the initial position at s_0. Curiously this is the same as the analogous *classical* result! No singular behavior occurs at the Schwarzschild radius, and the body falls continuously to $r = 0$ in a finite proper time, as shown in Fig. 6.2.

To describe the motion in terms of coordinate time t we form dr/dt from (6.180),

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = -c\sqrt{\frac{2m}{r}}\left(1 - \frac{2m}{r}\right) \tag{6.182}$$

and integrate to obtain

$$c(t_0 - t) = \frac{2}{3\sqrt{2m}}\left(r^{3/2} - r_0^{3/2} + 6m\sqrt{r} - 6m\sqrt{r_0}\right) - 2m \log \frac{(\sqrt{r} + \sqrt{2m})(\sqrt{r_0} - \sqrt{2m})}{(\sqrt{r_0} + \sqrt{2m})(\sqrt{r} - \sqrt{2m})} \tag{6.183}$$

This result is substantially different from (6.181). For situations where r and r_0 are much larger than $2m$ the two results are approximately the same, as should be expected, while for r very near $2m$ we have asymptotically

$$r - 2m = 8me^{-c(t-t_0)/2m} \tag{6.184}$$

It is thus apparent that $r = 2m$ is approached but never passed by the falling test body if one uses t as a time label. The nature of this radial motion is illustrated in Fig. 6.2 for a situation where r_0 is several times $2m$.

It is clear from the preceding that a description of the Schwarzschild geometry in terms of the coordinates r and t is limited: using proper time, we may study events that, in effect, occur *after* $t = \infty$. Certainly the coordinate t is very useful and physically meaningful since it corresponds to the proper time of an observer at rest far away from the central body. Thus in the finite proper time in which a test body falls to $r = 2m$ we would expect that the entire evolution of the physical universe exterior to $r = 2m$ has already occurred, so that the physical meaning of further fall becomes questionable in the context of Schwarzschild geometry. Nevertheless, we can still pose the purely mathematical question of how the fall proceeds in terms of coordinates that have a wider range of usefulness than the Schwarzschild coordinates. We shall therefore study the Kruskal coordinates in order to describe the entire Schwarzschild geometry conveniently.

To introduce Kruskal's coordinates we observe a peculiar feature of light propagation in terms of Schwarzschild coordinates. World-lines of light rays are characterized by a vanishing line element, $ds^2 = 0$. Thus for radial motion the path of a light ray is such that

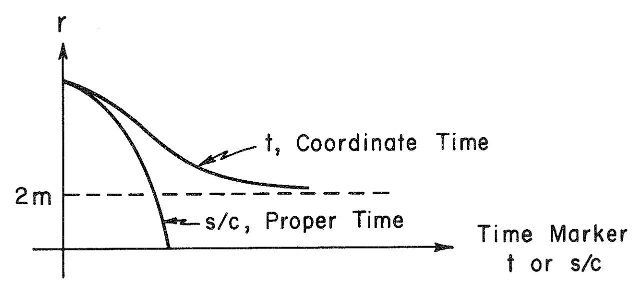
$$\left(1 - \frac{2m}{r}\right)c^2\,dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 = 0 \tag{6.185}$$

This means that the coordinate velocity of light is given by

$$\left(\frac{dr}{dt}\right)^2 = c^2\left(1 - \frac{2m}{r}\right)^2 \tag{6.186}$$

so that at $r = 2m$ the radial coordinate velocity of light becomes zero. This is an undesirable feature of the Schwarzschild coordinates that we can eliminate as follows; we seek a transformation from r and t to new

Fig. 6.2
Fall toward the origin of a Schwarzschild geometry in terms of coordinate time t and proper time on the test body s/c.



variables u and v in which the line element has the form

$$ds^2 = f^2(u,v)(dv^2 - du^2) - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{6.187}$$

By the same procedure as above we find the radial coordinate velocity of light to be unity everywhere

$$\left(\frac{du}{dv}\right)^2 = 1 \tag{6.188}$$

so long as f^2 has no zeros. Thus in the u, v coordinates no natural boundary to light propagation can occur.

It is a simple task to obtain from (6.187) differential equations which lead to a transformation from r, t to u, v coordinates and a nonzero function f. The angular coordinates θ and φ will not be changed. The fundamental transformation equation for the metric tensor,

$$g_{\alpha\beta} = \frac{\partial\hat{x}^\mu}{\partial x^\alpha}\frac{\partial\hat{x}^\nu}{\partial x^\beta}\hat{g}_{\mu\nu} \tag{6.189}$$

and the line elements (6.53) and (6.187) lead to the following differential equations to be solved

$$\begin{aligned} 1 - \frac{2m}{r} &= f^2\left[\left(\frac{\partial v}{\partial x^0}\right)^2 - \left(\frac{\partial u}{\partial x^0}\right)^2\right] \\ -\left(1 - \frac{2m}{r}\right)^{-1} &= f^2\left[\left(\frac{\partial v}{\partial r}\right)^2 - \left(\frac{\partial u}{\partial r}\right)^2\right] \qquad x^0 = ct \\ 0 &= \frac{\partial u}{\partial x^0}\frac{\partial u}{\partial r} - \frac{\partial v}{\partial x^0}\frac{\partial v}{\partial r} \end{aligned} \tag{6.190}$$

Note that the signs of u and v are not determined by these equations. To simplify we introduce a new radial parameter ξ and a function $F(\xi)$ by

$$\begin{aligned} \xi &= r + 2m \log\left|\frac{r}{2m} - 1\right| \\ F(\xi) &= \frac{1 - 2m/r}{f^2(r)} \end{aligned} \tag{6.191}$$

We have here assumed that a function f may be found which depends only on r; this is a critical point since an infinite number of transformations could lead to the metric form (6.187), and only this assumption leads to the Kruskal form and also removes the coordinate singularity

at $r = 2m$. The relations (6.190) now simplify to

$$\left(\frac{\partial v}{\partial x^0}\right)^2 - \left(\frac{\partial u}{\partial x^0}\right)^2 = F(\xi) \tag{6.192a}$$

$$\left(\frac{\partial v}{\partial \xi}\right)^2 - \left(\frac{\partial u}{\partial \xi}\right)^2 = -F(\xi) \tag{6.192b}$$

$$\frac{\partial u}{\partial x^0}\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial x^0}\frac{\partial v}{\partial \xi} \tag{6.192c}$$

If we add Eqs. (6.192a) and (6.192b) and alternately add or subtract twice (6.192c), we obtain

$$\left(\frac{\partial v}{\partial x^0} + \frac{\partial v}{\partial \xi}\right)^2 = \left(\frac{\partial u}{\partial x^0} + \frac{\partial u}{\partial \xi}\right)^2 \tag{6.193a}$$

$$\left(\frac{\partial v}{\partial x^0} - \frac{\partial v}{\partial \xi}\right)^2 = \left(\frac{\partial u}{\partial x^0} - \frac{\partial u}{\partial \xi}\right)^2 \tag{6.193b}$$

Using a relative plus sign for the roots of (6.193a) and a relative minus sign for the roots of (6.193b), we then obtain two equations (if we were to use the same sign, the Jacobian of the transformation would vanish):

$$\frac{\partial v}{\partial x^0} = \frac{\partial u}{\partial \xi} \qquad \frac{\partial v}{\partial \xi} = \frac{\partial u}{\partial x^0} \tag{6.194}$$

which lead to

$$\frac{\partial^2 u}{\partial x^{02}} - \frac{\partial^2 u}{\partial \xi^2} = 0 \qquad \frac{\partial^2 v}{\partial x^{02}} - \frac{\partial^2 v}{\partial \xi^2} = 0 \tag{6.195}$$

Thus both u and v satisfy the wave equation in x^0 and ξ. [If we had chosen the opposite roots in (6.193), the same equation (6.195) would have resulted.]

The general solution of the wave equation is an arbitrary twice-differentiable function of $x^0 \pm \xi$. Thus the solutions to (6.194) and (6.195) are easily seen to be

$$\begin{aligned} v &= h(\xi + x^0) + g(\xi - x^0) \\ u &= h(\xi + x^0) - g(\xi - x^0) \end{aligned} \tag{6.196}$$

where h and g are to be determined. Now we substitute u and v from (6.196) back into Eqs. (6.192); Eq. (6.192c) is automatically satisfied, while (6.192*a*) and (6.192*b*) are equivalent and lead to

$$-4h'(\xi + x^0)g'(\xi - x^0) = F(\xi) \tag{6.197}$$

where a prime indicates differentiation with respect to the argument. This is a remarkable equation that will lead to solutions for h, g, and F that are unique up to unimportant constants.

So far we have made no restrictions on the range of r in our transformation. Now we must specify whether r is greater than or less than $2m$, since in the two regions we shall have different transformation functions which must be patched together at the boundary. We first consider $r \geq 2m$, in which case F is positive from (6.191). To solve (6.197) we differentiate with respect to ξ and x^0 to obtain

$$\frac{F'(\xi)}{F(\xi)} = \frac{h''(\xi + x^0)}{h'(\xi + x^0)} + \frac{g''(\xi - x^0)}{g'(\xi - x^0)} \tag{6.198a}$$

$$0 = \frac{h''(\xi + x^0)}{h'(\xi + x^0)} - \frac{g''(\xi - x^0)}{g'(\xi - x^0)} \tag{6.198b}$$

Thus

$$[\log F(\xi)]' = 2[\log h'(\xi + x^0)]' \tag{6.199}$$

We may treat ξ and $y \equiv \xi + x^0$ as independent variables, which implies that the two sides of (6.199) are functions of two independent variables and must both be equal to some constant η. Thus from (6.199) and (6.198*b*) we see that h, g, and F are exponential functions. We therefore write the solution to (6.197) as

$$h(y) = \tfrac{1}{2}e^{\eta y} \qquad g(y) = -\tfrac{1}{2}e^{\eta y} \qquad F(\xi) = \eta^2 e^{2\eta\xi} \tag{6.200}$$

where the arbitrary additive constants are chosen to be zero and the multiplicative constants to be $\frac{1}{2}$ for convenience. Note that the relative sign of h and g is negative, as dictated by $F > 0$. Now from (6.191), (6.196), and (6.200) we have the transformation

$$\begin{aligned} u &= \left(\frac{r}{2m} - 1\right)^{2m\eta} e^{\eta r} \cosh \eta x^0 \\ v &= \left(\frac{r}{2m} - 1\right)^{2m\eta} e^{\eta r} \sinh \eta x^0 \\ f^2 &= \frac{2m}{\eta^2 r}\left(\frac{r}{2m} - 1\right)^{1-4m\eta} e^{-2\eta r} \end{aligned} \tag{6.201}$$

It remains only to choose the arbitrary parameter η; to do this we demand that f^2 have no zero or singularity at $r = 2m$, which requires that $\eta = 1/4m$. (Then ds^2 will vanish *only* on the light cone.) The transformation is thus, finally,

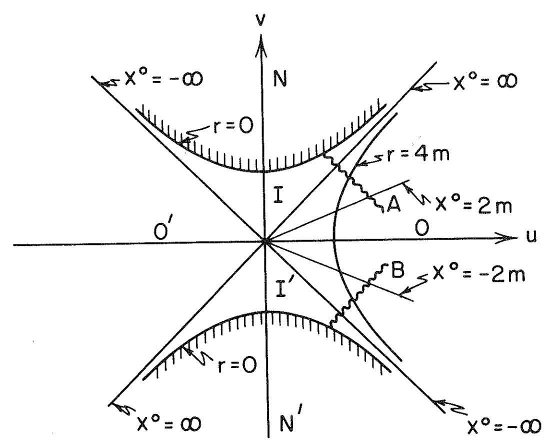
$$\begin{aligned} u &= \sqrt{\frac{r}{2m} - 1}\, e^{r/4m} \cosh \frac{x^0}{4m} \\ v &= \sqrt{\frac{r}{2m} - 1}\, e^{r/4m} \sinh \frac{x^0}{4m} \\ f^2 &= \frac{32m^3}{r} e^{-r/2m} \end{aligned} \tag{6.202}$$

The region of the uv plane defined by (6.202) for $r \geq 2m$ is $u \geq |v|$, which is labeled O in Fig. 6.3. Some special lines are of interest; for any finite x^0 the boundary line $r = 2m$ in the rt plane corresponds to the point $u = v = 0$ in the uv plane. Also we note that $x^0 \to \infty$ corresponds to $u = v$ and $x^0 \to -\infty$ corresponds to $u = -v$ for any value of $r > 2m$. For other points in O we invert the transformation (6.202)

$$u^2 - v^2 = \left(\frac{r}{2m} - 1\right) e^{r/2m} \qquad \frac{v}{u} = \tanh \frac{x^0}{4m} \tag{6.203}$$

Thus lines of constant r and lines of constant x^0 form a mesh of intersecting hyperbolas and rays in O as shown in Fig. 6.3. As r approaches

Fig. 6.3
Kruskal coordinates with several lines of constant r and t shown. The regions O and O' correspond to $r > 2m$, while I and I' correspond to $r < 2m$.



$2m$, the hyperbolas approach the lines $u = |v|$, but for $r = 2m$ the hyperbola degenerates to the point $u = v = 0$.

In Eq. (6.200) we arbitrarily chose h to be positive, so that g was negative. We could just as well have chosen the opposite signs, which would then have reversed the signs of both u and v in (6.202). We can therefore *identify* $-u$, $-v$ with u, v so that the nonoverlapping regions O and O' in the uv plane both represent the exterior of the Schwarzschild radius; we must still show that this is consistent with the transformation in the remainder of the uv plane.

For $r < 2m$ we must repeat some of our derivation. From (6.191) we see that F is negative, so that the relative sign of g and h must be positive. Proceeding as before with g and h both positive, we arrive at a transformation appropriate to the interior of the Schwarzschild radius

$$u = \sqrt{1 - \frac{r}{2m}}\, e^{r/4m} \sinh \frac{x^0}{4m}$$

(6.204)
$$v = \sqrt{1 - \frac{r}{2m}}\, e^{r/4m} \cosh \frac{x^0}{4m}$$

$$f^2 = \frac{32m^3}{r} e^{-r/2m}$$

This transformation relates $r < 2m$ to the region $v > |u|$, labeled I in Fig. 6.3. It is important to note that the transformations (6.204) and (6.202) match at the boundary $x^0 = \infty$ and, with appropriate sign changes, at $x^0 = -\infty$. The transformations (6.202) and (6.204) are therefore consistent. The inverse of (6.204) is

$$v^2 - u^2 = \left(1 - \frac{r}{2m}\right) e^{r/2m}$$

(6.205)
$$\frac{u}{v} = \tanh \frac{x^0}{4m}$$

so we have another mesh of hyperbolas and rays in I representing constant values of r and t. Note that the hyperbolas (6.203) and (6.205) are the *same*, and so our results for O and I are consistent as we approach the boundary $u = v$ from both sides. The origin $r = 0$ maps onto the hyperbola $v^2 - u^2 = 1$. As discussed for region O' we can identify points u, v in I' with $-u$, $-v$ in I. This is again consistent with values on the boundary curves.

In Kruskal's coordinates the geometry represented by the line element (6.187) with f^2 given in (6.202) is a solution of the Einstein equations and is nonsingular almost everywhere. Only along the hyperbola $v^2 - u^2 = 1$, corresponding to $r = 0$, do singularities occur in the Riemann tensor. Moreover, as we first demanded, light rays always travel along straight lines, $(du/dv)^2 = 1$, as in special relativity. Thus u serves as a global radial marker, and v serves as a global time marker. They do not, however, correspond to spherical coordinates for flat space at asymptotic distances, as the Schwarzschild coordinates do. We must also demand that Eq. (6.203) or (6.205) define r uniquely as an implicit function of u and v. This will be so as long as the right-hand side of (6.205) is a monotonic function of r. Since its derivative is $-(r/2m)e^{r/2m}$, it is monotonic for $r > 0$. Only at $r = 0$ does the derivative vanish and the above result break down; since this corresponds to $v^2 - u^2 = 1$, we see that the shaded regions N and N' in Fig. 6.3 must be deleted from the admissible regions of the Kruskal diagram.

The Schwarzschild metric is a nonsingular solution of the Einstein equation in O, O', I, and I' but has a singularity at the boundary corresponding to $r = 2m$. The Kruskal metric is also a nonsingular solution of the Einstein equations in these regions and is equivalent to the Schwarzschild solution, but it has no singularity at the boundary. This result is analogous to analytic continuation in complex-function theory and is appropriately referred to as an *analytic extension* of the manifold.

Let us use the Kruskal coordinates to study a light ray traveling radially inward toward the Schwarzschild radius, a problem we have already treated in Schwarzschild coordinates. Such a ray is represented by the line A in Fig. 6.3; it has slope -1. In terms of u, v the trajectory is simple; in terms of r and t, however, we see that it begins at some finite $r > 2m$ and finite x^0, travels inward toward $r = 2m$ as $x^0 \to \infty$, and *crosses* the line $x^0 = \infty$ to the interior of the Schwarzschild sphere. After that r continues to decrease along the trajectory, but x^0 *decreases*. This is in agreement with the behavior we previously obtained using Schwarzschild coordinates but goes beyond it and describes the trajectory subsequent to crossing the $x^0 = \infty$ line. The present treatment also clarifies the fact that x^0 is not a reasonable time marker inside the Schwarzschild sphere. For rays emitted from the interior of the Schwarzschild sphere the Kruskal coordinates remain useful. For example, consider a ray B, emitted from $r = 0$. It travels through increasing values of r but decreasing values of x^0, then crosses the line $x^0 = -\infty$ to the exterior, where its evolution is normal.

From the above it is clear that incoming light will in effect be totally absorbed by the Schwarzschild sphere. Since light emerging from the sphere must have been traveling since $x^0 = -\infty$, in effect before the beginning of time, it is questionable whether such light could be observed. If not, then the surface would have the physical aspect of a black hole, a surface that absorbs all light and emits none. It is evident also, a fortiori, that the same is true of massive bodies, since they move within the light cone, i.e., have a slope $(dv/du)^2 > 1$. In Chap. 7 we shall give a more general discussion of the nature and operation of this "one-way membrane."

We have discussed an ideal Schwarzschild black hole. During most of its lifetime the size of a star is determined by a balance between the inward pull of gravity and the pressure due to radiation released by the nuclear-fusion reaction in the deep interior. When the necessary light nuclei have been used up, the fusion process ceases, the stellar equilibrium cannot be maintained, and in some cases the gravitational force collapses the star, shrinking it to a size asymptotically approaching its Schwarzschild radius. Thus for radii slightly greater than $2m$ and times that are finite the geometry becomes that which we have discussed: the asymptotically collapsing star would appear as a black hole. Note that in this case the line $x^0 = -\infty$ is not part of the accessible exterior and photons such as B in Fig. 6.3 would certainly not be expected to be emitted by a real star approaching the black-hole state asymptotically.

Clearly our considerations apply only to nonrotating stars. In the next chapter we discuss black holes with rotation, and in Chap. 14 we discuss terminal stellar evolution and the formation of black holes.

Exercises

6.1 Verify that the off-diagonal elements of the contracted Riemann tensor for a Schwarzschild *form* of line element (6.9) are identically zero, as noted following Eq. (6.52).

6.2 What is $\sqrt{-g}$ for the Schwarzschild metric? Where is it singular?

6.3 Calculate the Riemann tensor for the Schwarzschild metric. Where does it have singularities? What are the components $R^i{}_{0k0}$, which correspond to derivatives of the Newtonian force in (5.147), and where are they singular? (These force derivatives correspond to *tidal forces*.)

6.4 Calculate the invariant $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ for the Schwarzschild metric. Where is it singular? Calculate the scalar curvature for the three-space part of the Schwarzschild metric, $R^i{}_i$. Where is it singular?

6.5 Verify that $\xi_\alpha = (1,0,0,0)$ is a Killing vector in the Schwarzschild geometry. What are the Killing vectors that correspond to rotational symmetry?

6.6 Show that the Petrov type of the Schwarzschild metric is *ID*. (This calculation is much easier than might be expected; why?)

6.7 Show that the effect of bending of the trajectory on the time delay for the radar-echo experiment is negligible.

6.8 Suppose that light were composed of corpuscules that behave like ordinary Newtonian particles in a classical gravitational field. They would then be distinguished only by their asymptotic velocity c in the region far from gravitational influence. Show that with this model one obtains one-half the Einstein value for the deflection of starlight by the sun.

6.9 With the Newtonian corpuscular model of Exercise 6.8 calculate the radar-echo time delay and show that one obtains minus one-half the general relativity result. Why is the sign different from the relativistic result?

6.10 Consider a Newtonian corpuscule projected radially outward from a classical point-particle field; let it begin with velocity v at radius r. If the particle has exactly the velocity necessary for escape, what is the relation between v and r? If the initial velocity is c, what is r? (This radius may be interpreted as the critical size of a Newtonian star from which light could not escape; it was discussed by Laplace in the late eighteenth century!)

6.11 Verify the theoretical values of the perihelia shifts in Table 6.1, in particular that of Icarus.

6.12 Discuss radar time delay along a radial path, e.g., between earth and Mercury at inferior conjunction.

Problems

6.1 Consider the functions λ and ν in Eq. (6.9) to be functions of both r and t. Evaluate the necessary Christoffel symbols, and show that

$$R_{01} = \frac{\partial\lambda/\partial t}{cr}.$$

Then show that the Einstein equations give $\lambda(r) + \nu(r,t) = k(t)$. Show that a change of the time coordinate similar to that in the text leads to the usual Schwarzschild solution. This is known as *Birkhoff's theorem*. It shows that the demand of time-independence in the Schwarzschild solution is superfluous (Birkhoff, 1923).

6.2 Modify the Schwarzschild solution so that $g_{00} = 1 - 2m/r$ and $g_{11} = (1 - 2\bar{m}/r)^{-1}$, where m and $\bar{m}$ may be different. Repeat the calculation of the deflection of starlight leading to (6.152) and show that the deflection is proportional to $\bar{m} + m$. That is, g_{00} and g_{11} contribute one-half each to the net deflection.

6.3 Consider a metric of the general spherically symmetric form (6.9) with the coefficients written as power series in m/r, a small quantity for celestial mechanics,

$$e^{\nu} = 1 + \alpha \frac{2m}{r} + \beta \frac{2m^2}{r^2} + \cdots$$

$$e_{\lambda} = 1 + \gamma \frac{2m}{r} + \delta \frac{2m^2}{r^2} + \cdots$$

What are the values of α, β, γ, etc., for the Schwarzschild metric? To see which terms of the Schwarzschild metric are actually tested by observation repeat the calculation of the four tests discussed in the text using this metric (Schiff, 1967).

6.4 Beginning at (6.192), introduce new dependent variables $u = H + G$, $v = H - G$ and new independent variables $y = \xi + x^0$, $z = \xi - x^0$. Show that H must be a function of y (or z) alone and G must be a function of z (or y) alone; thereby rederive Eq. (6.196).

6.5 Study the radial fall of a massive test body in Kruskal coordinates, using the standard Euler-Lagrange approach. Compare to the fall in Schwarzschild coordinates, and sketch the trajectory in the uv plane. How does a typical trajectory compare with a light-ray trajectory?

6.6 As introduced in the text, the concept of static metric involves a particular choice of coordinates. By considering a family of spacelike hypersurfaces and timelike curves orthogonal to them (hypersurface orthogonal) give a geometric and invariant characterization of a static geometry. Contrast with a geometry which is merely stationary, as discussed in Chap. 3 (see Vishveshwara, 1968, and Sec. 8.6).

6.7 In the text we studied the effect of a quadrupole moment on equatorial orbits and perihelion shifts. Study the effect on nonequatorial orbits and show that the relativistic effect and quadrupole effect are readily separated as a function of the angle between the orbital plane and the equatorial plane.

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CHAPTER

7

The Kerr Solution

The Schwarzschild solution, which we studied in the last chapter, describes the gravitational field exterior to a spherically symmetric body. It was obtained in 1916 and has long been a very useful model for problems in celestial mechanics and astrophysics. The problem of the field of a rotating spherical body was treated with perturbation theory by Lense and Thirring in 1918, but due to the complexity of the gravitational field equations no analogous exact solution was found for many years. In 1963 R. P. Kerr succeeded in obtaining an exact solution that represents the field exterior to a rotating axially symmetric body. The Kerr solution possesses interesting features in the region of very strong fields which are not present in the approximate Lense-Thirring solution and is therefore of particular interest in the study of the gravitational collapse of rotating stars.

In this chapter we shall derive the Kerr solution in a way that generalizes the Schwarzschild solution and discuss its physical features. We start by developing the algebraic groundwork necessary to simplify the field equations for the special metric form we shall use.

7.1 Eddington's Form of the Schwarzschild Solution

A. S. Eddington, in 1924, obtained a useful form of the Schwarzschild solution, upon which we shall base our derivation of the Kerr solution. The Schwarzschild solution is put into Eddington's form by a coordinate transformation containing a new time marker

$$(7.1)\quad \bar{x}^0 = x^0 + 2m \log \left| \frac{r}{2m} - 1 \right| \qquad \bar{r} = r \qquad \bar{\theta} = \theta \qquad \bar{\varphi} = \varphi$$

By a straightforward transformation of the metric tensor we find that the Schwarzschild line element with a barred time coordinate is

$$(7.2)\quad ds^2 = (d\bar{x}^0)^2 - (dr)^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - \frac{2m}{r}(d\bar{x}^0 + dr)^2$$

This, Eddington's form, is a flat-space line element plus a term with interesting properties. In Cartesian coordinates it is

$$(7.3)\quad \begin{aligned} ds^2 &= (d\bar{x}^0)^2 - (d\mathbf{x})^2 - \frac{2m}{r}\left(d\bar{x}^0 + \frac{x\,dx + y\,dy + z\,dz}{r}\right)^2 \\ (d\mathbf{x})^2 &= dx^2 + dy^2 + dz^2 \qquad r^2 = x^2 + y^2 + z^2 \end{aligned}$$

Thus the metric in Cartesian coordinates is

$$(7.4a)\quad g_{\mu\nu} = \eta_{\mu\nu} - 2ml_\mu l_\nu$$

$$(7.4b)\quad l_\mu = \frac{1}{\sqrt{r}}\left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

where $\eta_{\mu\nu}$ is the Lorentz metric and $l_\mu l_\nu \eta^{\mu\nu} = 0$.

The point of this exercise is the consideration of the *algebraic form* of the metric in (7.4*a*). We shall refer to this as a *degenerate* metric, for reasons to be discussed in the following sections.

7.2 Einstein's Equations for Degenerate Metrics

The preceding section serves as motivation for considering metrics of the general form

$$(7.5)\quad g_{\mu\nu} = \eta_{\mu\nu} - 2ml_\mu l_\nu \qquad l_\mu l_\nu \eta^{\mu\nu} = 0 \qquad m = \text{arbitrary constant}$$

It can be shown (see Prob. 7.1) that a metric of this form does not correspond to a Petrov type I Riemann tensor; we refer to such a space as *algebraically special* or *degenerate.* For brevity we therefore call the metric form (7.5) *degenerate;* it should be stressed, however, that not all algebraically special Riemann tensors correspond to a metric of the form (7.5) (see Prob. 7.7). As we shall see in this section and the next, the degenerate metric form greatly facilitates the purely algebraic simplification and solution of the Einstein equations.

Degenerate metrics have very interesting properties. Define a four-component upper-indexed object l^α by

$$(7.6)\quad l^\alpha \equiv \eta^{\alpha\tau} l_\tau$$

It is then easy to show that the matrix

$$(7.7)\quad g^{\mu\nu} = \eta^{\mu\nu} + 2ml^\mu l^\nu$$

is the inverse of $g_{\mu\nu}$ and therefore is the contravariant metric tensor. From this it follows that

$$(7.8)\quad l^\alpha = g^{\alpha\tau} l_\tau = \eta^{\alpha\tau} l_\tau$$

so that l^α is actually the contravariant four-vector corresponding to l_μ; its indices may be raised and lowered with either the true metric or the Lorentz metric. The vector l_μ has other interesting properties; since it is null by assumption,

$$(7.9)\quad l^\alpha l_{\alpha|\tau} = l_\alpha l^\alpha{}_{|\tau} = \tfrac{1}{2}(\eta^{\mu\nu} l_\mu l_\nu)_{|\tau} = 0$$

By enumeration it is easy to show that

$$(7.10)\quad \left\{ {\alpha \atop \beta\ \mu} \right\} l^\mu = -ml^\nu (l^\alpha l_\beta)_{|\nu}$$

from which we see that the covariant form of (7.9) also holds

$$(7.11)\quad l_\nu l^\nu{}_{\|\alpha} = l^\nu l_{\nu\|\alpha} = l^\nu \left(l_{\nu|\lambda} - \left\{ {\tau \atop \nu\ \lambda} \right\} l_\tau \right) = l^\nu l_{\nu|\lambda} = 0$$

The Einstein field equations are also simplified by our choice of metric. Consider first the metric determinant g. At any point l_μ is a flat-space null vector; that is, $l_\mu l_\nu \eta^{\mu\nu} = 0$. We can perform a proper rotation of coordinates in three-space that leaves $\eta^{\mu\nu}$ invariant and brings l_μ into the form $(a,a,0,0)$. In this system we have

$$(7.12)\quad g = \begin{Vmatrix} 1 - 2ma^2 & -2ma^2 & 0 & 0 \\ -2ma^2 & -1 - 2ma^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} = -1$$

But from the discussion in Sec. 3.5 we know that g will behave like a scalar under a three-dimensional rotation, since the Jacobian of such a transformation is 1. Thus, $g = -1$ for any degenerate metric. From this fact and Eq. (3.11) it follows that

$$\left\{ {\alpha \atop \rho\ \alpha} \right\} = \frac{\partial}{\partial x^\rho} \log \sqrt{-g} = 0 \tag{7.13}$$

Thus the field equations (5.119) are simple and contain only two terms

$$R_{\mu\nu} = -\left\{ {\alpha \atop \mu\ \nu} \right\}_{|\alpha} + \left\{ {\alpha \atop \beta\ \mu} \right\} \left\{ {\beta \atop \alpha\ \nu} \right\} = 0 \tag{7.14}$$

Because $g_{\mu\nu}$ is a polynomial in m, so is the Ricci tensor: $g_{\mu\nu}$ is first order and $R_{\mu\nu}$ is fourth order. Moreover, m is *arbitrary*, so that $g_{\mu\nu}$ must be a solution for *any* value of m. Thus, in the expression for $R_{\mu\nu}$ as a polynomial in m, each order must vanish separately. As a result, we have four sets of 10 equations to solve, which oddly enough will simplify our task. If we note that the Christoffel symbol of the first kind, $[\alpha\beta,\gamma]$, is linear in m, power counting is easy and the four sets of equations are

$$\eta^{\alpha\rho}[\mu\nu,\rho]_{|\alpha} = 0 \qquad O(m) \tag{7.15a}$$

$$2m(l^\alpha l^\rho[\mu\nu,\rho])_{|\alpha} - \eta^{\alpha\sigma}\eta^{\beta\lambda}[\beta\mu,\sigma][\alpha\nu,\lambda] = 0 \qquad O(m^2) \tag{7.15b}$$

$$l^\beta l^\lambda \eta^{\alpha\sigma}[\beta\mu,\sigma][\alpha\nu,\lambda] + l^\alpha l^\lambda \eta^{\beta\sigma}[\beta\mu,\lambda][\alpha\nu,\sigma] = 0 \qquad O(m^3) \tag{7.15c}$$

$$l^\alpha l^\sigma l^\beta l^\lambda[\beta\mu,\sigma][\alpha\nu,\lambda] = 0 \qquad O(m^4) \tag{7.15d}$$

It is easy to show by explicitly writing out terms that the order m^4 equations (7.15*d*) are satisfied identically. Similarly the order m^3 equations lead to

$$-ml_\mu l_\nu(v^\alpha v_\alpha) = 0 \qquad v^\alpha \equiv l^\beta l^\alpha{}_{\|\beta} = l^\beta l^\alpha{}_{|\beta} \tag{7.16}$$

Thus v^α is a null vector. Moreover, it is orthogonal to the null vector l^α, as is easily seen:

$$v^\nu l_\nu = (l^\alpha l^\nu{}_{|\alpha})l_\nu = l^\alpha(l_\nu l^\nu{}_{|\alpha}) = 0 \tag{7.17}$$

From this it is easy to show that v^α and l^α must be proportional. To do this we first note that the indices of v^α may be raised and lowered with the Lorentz metric, as with l^α. At any chosen point l^μ and v^μ may be written (**l** and **v** are ordinary three-vectors in Euclidean space)

$$l^\nu = (|\mathbf{l}|,\mathbf{l}) \qquad v^\nu = (|\mathbf{v}|,\mathbf{v}) \tag{7.18}$$

since they are null with respect to $\eta_{\mu\nu}$. Because they are orthogonal,

$$l^\nu v_\nu = |\mathbf{l}||\mathbf{v}|(1 - \cos\theta) = 0 \qquad \cos\theta = \frac{\mathbf{l}\cdot\mathbf{v}}{|\mathbf{l}|\cdot|\mathbf{v}|} \tag{7.19}$$

Thus $\cos\theta = 1$, and **v** is parallel to **l** at any given point. We may therefore write

$$v^\nu = l^\alpha l^\nu{}_{|\alpha} = -A(x^\mu)l^\nu \tag{7.20}$$

where A is a scalar field.

We shall defer discussion of the order m^2 equations till later, when we shall show that they are identically satisfied.

The order m equations can be further simplified. From (7.15*a*) they are

$$\eta^{\alpha\rho}[(l_\nu l_\rho)_{|\mu|\alpha} + (l_\rho l_\mu)_{|\nu|\alpha} - (l_\mu l_\nu)_{|\rho|\alpha}] = 0 \tag{7.21}$$

If we introduce the D'Alembertian operator, $\Box^2 = (\partial^2/\partial x^{0^2}) - \nabla^2$, and a scalar L defined by [we use (7.13)]

$$L = -l^\alpha{}_{\|\alpha} = -\left(l^\alpha{}_{|\alpha} + \left\{ {\alpha \atop \alpha\ \tau} \right\} l^\tau\right) = -l^\alpha{}_{|\alpha} \tag{7.22}$$

these simplify to

$$-\Box^2(l_\mu l_\nu) = [(L + A)l_\mu]_{|\nu} + [(L + A)l_\nu]_{|\mu} \tag{7.23}$$

We have so far not assumed any symmetry of the metric. In Sec. 7.4 we shall specialize to the case where the metric is stationary, or independent of x^0. A considerable simplification of the order m equations (7.23) will result.

7.3 The Order m^2 Equations

In this brief section we shall demonstrate that the order m^2 equations (7.15*b*) reduce to a scalar equation which is automatically satisfied by any solution of the order m equations (7.23). It is slightly tedious but elementary to write out all the terms indicated in (7.15*b*) and use (7.5), (7.9), (7.16), and (7.20) to simplify. The result is

$$l_\mu l_\nu[2(l^\alpha A)_{|\alpha} - A^2 + l^\alpha{}_{|\beta}l^\beta{}_{|\alpha} - l^\alpha{}_{|\beta}l_\alpha{}^{|\beta}] = 0 \qquad l_\alpha{}^{|\beta} \equiv \eta^{\beta\tau}l_{\alpha|\tau} \tag{7.24}$$

This implies the scalar equation

$$2(l^\alpha A)_{|\alpha} - A^2 + l^\alpha{}_{|\beta}l^\beta{}_{|\alpha} - l^\alpha{}_{|\beta}l_\alpha{}^{|\beta} = 0 \tag{7.25}$$

We first simplify the third term, using (7.20) and (7.22):

$$l^\alpha{}_{|\beta}l^\beta{}_{|\alpha} = (l^\alpha{}_{|\beta}l^\beta)_{|\alpha} - l^\alpha{}_{|\beta|\alpha}l^\beta = -(Al^\alpha)_{|\alpha} + L_{|\beta}l^\beta = [(L - A)l^\alpha]_{|\alpha} + L^2 \tag{7.26}$$

Similarly the fourth term can be simplified using (7.9):

$$l^\alpha{}_{|\beta}l_\alpha{}^{|\beta} = (l^\alpha l_\alpha{}^{|\beta})_{|\beta} - l^\alpha l_\alpha{}^{|\beta}{}_{|\beta} = -l^\alpha l_\alpha{}^{|\beta}{}_{|\beta} \tag{7.27}$$

Manipulation of the order m equations will give a simpler form yet for this expression; expanding (7.23) gives

$$-l_\mu l_\nu{}^{|\beta}{}_{|\beta} - l_\nu l_\mu{}^{|\beta}{}_{|\beta} - 2l_\mu{}^{|\beta}l_{\nu|\beta} = (L + A)_{|\mu}l_\nu + (L + A)_{|\nu}l_\mu + (L + A)(l_{\mu|\nu} + l_{\nu|\mu}) \tag{7.28}$$

When contracted with l^μ, this implies

$$-l_\nu l^\mu l_\mu{}^{|\beta}{}_{|\beta} = (L + A)_{|\mu}l^\mu l_\nu + (L + A)l_{\nu|\mu}l^\mu = l_\nu l^\mu(L + A)_{|\mu} - (L + A)Al_\nu \tag{7.29}$$

The common factor l_ν cancels, and we get

$$\begin{aligned} -l^\mu l_\mu{}^{|\beta}{}_{|\beta} &= l^\mu(L + A)_{|\mu} - (L + A)A \\ &= [(L + A)l^\mu]_{|\mu} - l^\mu{}_{|\mu}(L + A) - A(L + A) \\ &= [(L + A)l^\mu]_{|\mu} + L^2 - A^2 \end{aligned} \tag{7.30}$$

and thus the fourth term of (7.25) can be written

$$l^\alpha{}_{|\beta}l_\alpha{}^{|\beta} = L^2 - A^2 + [(L + A)l^\mu]_{|\mu} \tag{7.31}$$

We can now substitute the above results (7.26) and (7.31) into the left side of (7.25), to obtain

$$2(l^\alpha A)_{|\alpha} - A^2 + [(L - A)l^\mu]_{|\mu} + L^2 - (L^2 - A^2) - [(L + A)l^\mu]_{|\mu} = 2(Al^\alpha)_{|\alpha} - 2(Al^\mu)_{|\mu} = 0 \tag{7.32}$$

That is, Eq. (7.25) is satisfied identically. The entire content of the field equations is thus embodied in Eq. (7.23).

7.4 Field Equations for the Stationary Case

In the stationary, or time-independent, case it is possible to reduce the field equations to two simple partial differential equations for a single complex function. The simplification is achieved by elementary algebraic manipulation of the order m equations (7.23).

We begin by introducing a three-vector λ_j via the equation

$$l_\mu = l_0(1,\lambda_1,\lambda_2,\lambda_3) \tag{7.33}$$

Since l_μ is a flat-space null vector ($l_\mu l_\nu\eta^{\mu\nu} = 0$), λ_j is a flat-space unit vector, $\boldsymbol{\lambda}^2 = 1$. Expressed in terms of λ_j, the order m equations are

$$\nabla^2(l_0^2) = 0 \qquad \mu = \nu = 0 \tag{7.34a}$$

$$\nabla^2(l_0^2\lambda_j) = [(L + A)l_0]_{|j} \qquad \mu = 0 \qquad \nu = j \neq 0 \tag{7.34b}$$

$$\nabla^2(l_0^2\lambda_i\lambda_j) = [(L + A)l_0\lambda_i]_{|j} + [(L + A)l_0\lambda_j]_{|i} \qquad \mu = i \neq 0 \qquad \nu = j \neq 0 \tag{7.34c}$$

and our task is to obtain l_0 and λ_j. We can manipulate these equations to obtain a first-order differential equation to replace (7.34*c*), which represents a considerable simplification. To do this we expand (7.34*c*), using (7.34*a*) to discard terms; (7.34*b*) then allows us to cancel all the second-order terms. This leaves

$$\lambda_{i|j} + \lambda_{j|i} = \frac{2l_0}{L + A}\lambda_{i|k}\lambda_{j|k} \equiv \frac{1}{p}\lambda_{i|k}\lambda_{j|k} \tag{7.35}$$

In *this* section we shall sum over any repeated index, regardless of position; for example, k is to be summed over. The gravitational field is now described by equations (7.34*a*), (7.34*b*), and (7.35).

We shall solve (7.35) for $\lambda_{i|j}$ as a function of λ_k, with only one arbitrary parameter. To do this we denote the 3×3 matrix $\lambda_{i|j}$ by M, so that (7.35) may be written in matrix notation as

$$M + M^T = \frac{1}{p} M M^T \tag{7.36}$$

The constant length of λ_j, $\boldsymbol{\lambda}^2 = 1$, leads to an important matrix equation

$$\tfrac{1}{2}(\lambda_i \lambda_i)_{|k} = \lambda_i \lambda_{i|k} = 0 \qquad M^T \boldsymbol{\lambda} = 0 \tag{7.37}$$

That is, $\boldsymbol{\lambda}$ is in the null space of M^T. Moreover, (7.20) with $\nu = 0$ gives

$$\lambda_i l_{0|i} = A \tag{7.38}$$

and with $\nu = k \neq 0$

$$l_0 \lambda_j \lambda_{k|j} + \lambda_k \lambda_j l_{0|j} = A \lambda_k \tag{7.39}$$

Combining these, we arrive at

$$\lambda_{k|j} \lambda_j = 0 \qquad M \boldsymbol{\lambda} = 0 \tag{7.40}$$

so $\boldsymbol{\lambda}$ is *also* in the null space of M. Using only matrix algebra, we now proceed to solve (7.36), (7.37), and (7.40) for M as a function of $\boldsymbol{\lambda}$.

The nonlinear equation (7.36) may be reduced in dimension with the aid of (7.37) and (7.40). At any given point choose a 3×3 rotation matrix R that brings $\boldsymbol{\lambda}$ onto the x axis. That is,

$$R\boldsymbol{\lambda} = \boldsymbol{\lambda}' \qquad \boldsymbol{\lambda}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{7.41}$$

If $\boldsymbol{\lambda}$ is in the null space of M and M^T, then $\boldsymbol{\lambda}'$ is in the null space of M' and M'^T, where

$$M' = RMR^T \qquad M'^T = RM^TR^T \tag{7.42}$$

From the explicit form of $\boldsymbol{\lambda}'$ in (7.41) and the fact that $\boldsymbol{\lambda}'$ is in the null space of M' and M'^T we see that M' must have the form

$$M' = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{N'}} \\ 0 & & \end{array}\right) \qquad N' \text{ is } 2 \times 2 \tag{7.43}$$

Moreover, since matrix algebra is invariant under rotations, the matrices M' and N' may be easily seen to also obey (7.36)

$$N' + N'^T = \frac{1}{p} N'N'^T \tag{7.44}$$

This is a well-known condition related to unitary matrices; one can easily see that it is equivalent to the statement that the real matrix $I - (1/p)N'$ is unitary, i.e.,

$$U = I - \frac{1}{p} N' \qquad UU^T = U^TU = I \tag{7.45}$$

Since N' is 2×2 and real, it is therefore either a proper rotation in two dimensions or an improper rotation, i.e., a rotation plus inversion. Thus it may be written

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$$
$$|U| = \begin{cases} 1 & \text{proper case} \\ -1 & \text{improper case} \end{cases} \tag{7.46}$$

We shall work only with the proper rotation matrix; our justification lies in the interesting results that follow from it. We now see from (7.45) and (7.43) that N' and M' are

$$N' = p \begin{pmatrix} 1 - \cos\theta & \sin\theta \\ -\sin\theta & 1 - \cos\theta \end{pmatrix}$$
$$M' = p \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - \cos\theta & \sin\theta \\ 0 & -\sin\theta & 1 - \cos\theta \end{pmatrix} \tag{7.47}$$

We must now rotate back to the original coordinates to get $M = R^TM'R$. The simple form assumed by M' allows us to write

$$\begin{aligned} M_{ik} &= R_{li} R_{jk} M'_{lj} \\ &= p(1 - \cos\theta)(R_{2i}R_{2k} + R_{3i}R_{3k}) \\ &\qquad + p \sin\theta (R_{2i}R_{3k} - R_{3i}R_{2k}) \end{aligned} \tag{7.48}$$

But since R is a rotation matrix with orthonormal rows and columns, this condenses. The orthonormality of the columns and the fact that the columns form a right-handed triad of vectors imply

$$R_{1i}R_{1k} + R_{2i}R_{2k} + R_{3i}R_{3k} = \delta_{ik}$$
$$R_{2i}R_{3k} - R_{3i}R_{2k} = \epsilon_{ikl}R_{1l} \tag{7.49}$$

Thus M_{ik} may be expressed in terms of θ and only the first row of R, $R_{1i} \equiv R_i$:

$$M_{ik} = p(1 - \cos\theta)(\delta_{ik} - R_iR_k) + p\sin\theta\,\epsilon_{ikl}R_l \tag{7.50}$$

To complete our construction of M in terms of $\boldsymbol{\lambda}$ and θ we recall from (7.41) that the matrix R rotates $\boldsymbol{\lambda}$ into $\boldsymbol{\lambda}'$, $\boldsymbol{\lambda}' = (1,0,0)$. Thus we must have, by expressing the first element of (7.41) in vector notation,

$$\mathbf{R}\cdot\boldsymbol{\lambda} = 1 \tag{7.51}$$

But $\mathbf{R}$ and $\boldsymbol{\lambda}$ are both unit vectors, and so the angle α between them must be zero,

$$\mathbf{R}\cdot\boldsymbol{\lambda} = \cos\alpha = 1 \qquad \alpha = 0 \tag{7.52}$$

and hence $\mathbf{R} = \boldsymbol{\lambda}$ or $R_i = \lambda_i$. Then we have completed our task and may express M as

$$M_{ik} = \lambda_{i|k} = p(1 - \cos\theta)(\delta_{ik} - \lambda_i\lambda_k) + p\sin\theta\,\epsilon_{ikl}\lambda_l \tag{7.53}$$

This is a most useful result; it replaces the nonlinear implicit relation (7.35) by a simple explicit expression for $\lambda_{i|k}$. Our further development rests very heavily on this equation.

A consideration of the algebraic content of (7.53) will lead to the two simple partial differential equations mentioned at the beginning of this section. We first rewrite (7.53) with parameters α and β replacing p and θ:

$$\lambda_{i|k} = \alpha(\delta_{ik} - \lambda_i\lambda_k) + \beta\epsilon_{ikl}\lambda_l \tag{7.54}$$

It will turn out that α and β determine the metric in a very elegant manner. A number of simple and important three-vector relations follow directly from (7.54). Setting $i = k$ and summing gives

$$\nabla\cdot\boldsymbol{\lambda} = 2\alpha \tag{7.55}$$

Multiplying by ϵ_{jki} and summing over i and k gives

$$\nabla\times\boldsymbol{\lambda} = -2\beta\boldsymbol{\lambda} \tag{7.56}$$

The Laplacian of $\boldsymbol{\lambda}$ may be obtained in two ways. Differentiating (7.54) with respect to x^k gives

$$\nabla^2\boldsymbol{\lambda} = \nabla\alpha - \boldsymbol{\lambda}(\nabla\alpha\cdot\boldsymbol{\lambda}) - 2(\alpha^2 + \beta^2)\boldsymbol{\lambda} + \nabla\beta\times\boldsymbol{\lambda} \tag{7.57}$$

Alternatively, from (7.56) and a well-known vector identity

$$\nabla\times(\nabla\times\boldsymbol{\lambda}) = \nabla(\nabla\cdot\boldsymbol{\lambda}) - \nabla^2\boldsymbol{\lambda} = -2(\nabla\times\beta\boldsymbol{\lambda}) \tag{7.58}$$

Solving for $\nabla^2\boldsymbol{\lambda}$ and using (7.55) and (7.56) to simplify, we then have a second form

$$\nabla^2\boldsymbol{\lambda} = 2\nabla\alpha + 2(\nabla\beta\times\boldsymbol{\lambda}) - 4\beta^2\boldsymbol{\lambda} \tag{7.59}$$

Equating the expressions (7.57) and (7.59) for $\nabla^2\boldsymbol{\lambda}$, we obtain

$$\nabla\alpha = -\nabla\beta\times\boldsymbol{\lambda} - \boldsymbol{\lambda}(\nabla\alpha\cdot\boldsymbol{\lambda}) - 2(\alpha^2 - \beta^2)\boldsymbol{\lambda} \tag{7.60}$$

This leads finally to the following very important equations:

$$\nabla\alpha\cdot\boldsymbol{\lambda} = \beta^2 - \alpha^2 \tag{7.61a}$$

$$\nabla\alpha = (\beta^2 - \alpha^2)\boldsymbol{\lambda} - \nabla\beta\times\boldsymbol{\lambda} \tag{7.61b}$$

Equations analogous to (7.61) with β replacing α on the left-hand side can be obtained. From (7.56) the divergence of $\beta\boldsymbol{\lambda}$ is zero, so that

$$\nabla\cdot\beta\boldsymbol{\lambda} = \beta(\nabla\cdot\boldsymbol{\lambda}) + \nabla\beta\cdot\boldsymbol{\lambda} \tag{7.62}$$

and thus, from (7.55),

$$\nabla\beta\cdot\boldsymbol{\lambda} = -2\alpha\beta \tag{7.63}$$

analogous to (7.61*a*). Now we cross $\boldsymbol{\lambda}$ with (7.61*b*), simplify, and solve for $\nabla\beta$:

$$\nabla\beta = \boldsymbol{\lambda}(\boldsymbol{\lambda}\cdot\nabla\beta) + (\nabla\alpha\times\boldsymbol{\lambda}) \tag{7.64}$$

or, with the use of (7.63),

$$\nabla\beta = -2\alpha\beta\boldsymbol{\lambda} + (\nabla\alpha \times \boldsymbol{\lambda}) \tag{7.65}$$

analogous to (7.61*b*).

Equations (7.61), (7.63), and (7.65) are very important and may be expressed in more concise fashion by introducing a complex function $\gamma = \alpha + i\beta$:

$$\nabla\gamma \cdot \boldsymbol{\lambda} = -\gamma^2 \tag{7.66a}$$

$$\nabla\gamma = -\gamma^2\boldsymbol{\lambda} + i(\nabla\gamma \times \boldsymbol{\lambda}) \tag{7.66b}$$

The importance of the introduction of γ should be stressed. It allows us to simplify the Einstein field equations greatly and will make the relation between the Kerr and Schwarzschild solutions transparent.

It remains in this section to obtain a pair of simple differential equations that determine γ and to show that γ in turn determines the metric, i.e., the functions l_0 and λ_j. The first differential equation is obtained by forming the Laplacian of γ from (7.66*b*) and using (7.55) and (7.56) to simplify

$$\begin{aligned} \nabla^2\gamma &= -\gamma^2\,\nabla\cdot\boldsymbol{\lambda} - 2\gamma\,\nabla\gamma\cdot\boldsymbol{\lambda} + i\,\nabla\cdot(\nabla\gamma\times\boldsymbol{\lambda}) \\ &= -2\alpha\gamma^2 + 2\gamma^3 - i\,\nabla\gamma\cdot(\nabla\times\boldsymbol{\lambda}) \\ &= -2\alpha\gamma^2 + 2\gamma^3 - 2i\beta\gamma^2 \\ &= -2\gamma^2(\alpha + i\beta - \gamma) \\ &= 0 \end{aligned} \tag{7.67}$$

Thus γ is a complex harmonic function. The second differential equation is obtained by squaring (7.66*b*) and using (7.66*a*) to simplify

$$(\nabla\gamma)^2 = \gamma^4 - (\nabla\gamma \times \boldsymbol{\lambda})^2 = \gamma^4 - ((\nabla\gamma)^2 - \gamma^4) = \gamma^4 \tag{7.68}$$

The last two differential equations determine γ. However, for maximum clarity we shall rewrite (7.68) in terms of the more convenient variable $\omega = 1/\gamma$ and repeat (7.67):

$$\nabla^2\gamma = 0 \qquad (\nabla\omega)^2 = 1 \qquad \omega \equiv \frac{1}{\gamma} \tag{7.69}$$

These are the Laplace and eikonal equations, familiar in classical optics. They determine the function γ completely, dependent upon consistent boundary conditions. More importantly, they completely replace the field equations in the various other forms in which we have written them, since, as we shall show, the metric functions l_0 and λ_j are determined by γ.

We can easily solve (7.66) for λ in terms of ω, which is more convenient than γ for this purpose. In terms of ω (7.66) reads

$$\boldsymbol{\lambda}\cdot\nabla\omega = \boldsymbol{\lambda}\cdot\nabla\omega^* = 1 \qquad \nabla\omega = \boldsymbol{\lambda} + i(\nabla\omega \times \boldsymbol{\lambda}) \tag{7.70}$$

Thus

$$\nabla\omega \times \nabla\omega^* = -i[\nabla\omega^* + \nabla\omega] + B\boldsymbol{\lambda} \tag{7.71}$$

where B represents a function of $\boldsymbol{\lambda}$ and ω which we need not write out. We can solve (7.71) for B by dotting $\nabla\omega$ into (7.71) and using (7.70):

$$B = i[1 + \nabla\omega\cdot\nabla\omega^*] \tag{7.72}$$

Now (7.71) is easily soluble for $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda} = \frac{\nabla\omega + \nabla\omega^* - i(\nabla\omega \times \nabla\omega^*)}{1 + \nabla\omega\cdot\nabla\omega^*} \tag{7.73}$$

It now remains only to obtain l_0 in terms of γ. The function l_0 must satisfy Eqs. (7.34*a*) and (7.34*b*). We shall show that

$$l_0^2 = \mathrm{Re}\,(\gamma) = \alpha \tag{7.74}$$

or a multiple of this is the unique solution of these equations consistent with (7.34*c*). From (7.69) we know that α is harmonic, so that (7.34*a*) is immediately satisfied. To show that $l_0^2 = \alpha$ is a solution of (7.34*b*) we first calculate the left-hand side using $l_0^2 = \alpha$ as a trial solution, with α harmonic

$$\nabla^2(\alpha\lambda_j) = \alpha\,\nabla^2\lambda_j + 2\alpha_{|k}\lambda_{j|k} \tag{7.75}$$

With the aid of (7.54) and (7.59) this becomes

$$\nabla^2(\alpha\lambda_j) = 4\alpha\,\nabla\alpha + 2\alpha(\alpha^2 + \beta^2)\boldsymbol{\lambda} + 2\alpha(\nabla\beta \times \boldsymbol{\lambda}) + 2\beta(\nabla\alpha \times \boldsymbol{\lambda}) \tag{7.76}$$

or, using (7.61*b*) and (7.65) to simplify further,

$$\nabla^2(\alpha\boldsymbol{\lambda}) = 2\alpha\,\nabla\alpha + 2\beta\,\nabla\beta = \nabla(\alpha^2 + \beta^2) \tag{7.77}$$

We next calculate the right-hand side of (7.34*b*). From the definitions $\alpha = p(1 - \cos\theta)$ and $\beta = p \sin\theta$ preceding (7.54) we find

$$\alpha^2 + \beta^2 = 2p^2(1 - \cos\theta) = 2\alpha p \tag{7.78}$$

From the definition of p in (7.35),

$$p = \frac{L + A}{2l_0} \tag{7.79}$$

We then have a relation between $L + A$ and α and β,

$$A + L = \frac{l_0}{\alpha}(\alpha^2 + \beta^2) \tag{7.80}$$

Thus with $l_0^2 = \alpha$ the right-hand side of (7.34*b*) is

$$[(L + A)l_0]_{|j} = (\alpha^2 + \beta^2)_{|j} \tag{7.81}$$

Comparing this with the right-hand side as obtained in (7.77), we see that $l_0^2 = \alpha$ is indeed a solution. It is moreover easily shown to be the unique solution, up to a multiplicative constant (see Exercise 7.3).

Let us now summarize the extensive simplification we have made. The field equations reduce to the pair of simple equations (7.69), with the metric functions given explicitly by (7.73) and (7.74). As we shall see in more detail, the complex function γ plays the role of a generalized Newtonian potential since it obeys Laplace's equation, and in the weak-field limit Re (γ) is precisely the Newtonian potential.

7.5 The Schwarzschild and Kerr Solutions

We now wish to solve the Einstein field equations for the stationary degenerate metric. These equations have now been greatly simplified to the form in Eq. (7.69), with (7.73) and (7.74) giving the metric functions explicitly. In analogy with Newtonian theory we first consider the simple spherically symmetric solution to Laplace's equation:

$$\gamma = \frac{1}{r} = [x^2 + y^2 + z^2]^{-1/2} \tag{7.82}$$

It is easily checked that $\omega = r$ satisfies the eikonal equation, so that (7.82) is a solution of the system (7.69). The metric function l_0^2, the analogue of the Newtonian potential, and the vector λ_i are then easily

obtained from (7.73) and (7.74)

$$l_0^2 = \frac{1}{r} \qquad \lambda_1 = \frac{x}{r} \qquad \lambda_2 = \frac{y}{r} \qquad \lambda_3 = \frac{z}{r} \tag{7.83}$$

Thus from the definitions (7.33) and (7.5) we obtain the line element

$$ds^2 = (dx^0)^2 - (d\mathbf{x})^2 - \frac{2m}{r}\left(dx^0 + \frac{x\,dx + y\,dy + z\,dz}{r}\right)^2 \tag{7.84}$$

This is precisely the Eddington form of the Schwarzschild solution discussed in Sec. 7.1. From this fact we can now identify the arbitrary parameter m as the geometric mass of the source.

We now ask for the simplest generalization of the Schwarzschild solution in the above context. We are naturally led to consider a general displacement of the origin, i.e.,

$$\gamma = \frac{1}{r} = [(x - a_1)^2 + (y - a_2)^2 + (z - a_3)^2]^{-1/2} \qquad a_i = \text{const} \tag{7.85}$$

since this satisfies (7.69) for any choice of the constants a_i. However, if the a_i are real, this solution corresponds to a physical displacement of the origin and is of no physical interest. On the other hand, an imaginary set of a_i represents a new physical situation. Without loss of generality (see Exercise 7.5) we may write such an imaginary displaced γ function as

$$\gamma = (x^2 + y^2 + (z - ia)^2)^{-1/2} \tag{7.86}$$

This represents the Kerr solution.

From this function we can obtain the metric functions l_0^2 and λ_i from (7.73) and (7.74), just as with the Schwarzschild case above, but with slightly more algebra. We first split ω into real and imaginary parts for convenience

$$\omega = \rho + i\sigma = (r^2 - a^2 - 2iaz)^{1/2} \qquad r^2 \equiv x^2 + y^2 + z^2 \tag{7.87}$$

Squaring this equation and equating real and imaginary parts, we get

$$\rho^2 - \sigma^2 = r^2 - a^2 \qquad \sigma = -\frac{az}{\rho} \tag{7.88}$$

from which we obtain a quadratic relation between ρ^2 and the Cartesian markers

$$\rho^4 - \rho^2(r^2 - a^2) - a^2z^2 = 0 \tag{7.89}$$

Explicitly ρ^2 is given by

$$\rho^2 = \frac{r^2 - a^2}{2} + \left[\frac{(r^2 - a^2)^2}{4} + a^2z^2\right]^{1/2} \tag{7.90}$$

Note that we have chosen the plus sign so that for $r \gg a$ the marker ρ is asymptotically equal to r, as is necessary according to (7.87). It is now easy to write out γ and obtain $l_0^2 = \text{Re}(\gamma)$ in a useful and simple form

$$\begin{aligned} \gamma &= \frac{1}{\rho + i\sigma} = \frac{\rho}{\rho^2 + \sigma^2} - \frac{i\sigma}{\rho^2 + \sigma^2} \\ l_0^2 &= \frac{\rho}{\rho^2 + \sigma^2} = \frac{\rho^3}{\rho^4 + a^2z^2} \end{aligned} \tag{7.91}$$

where we have used (7.88) to eliminate σ.

To obtain the vector $\boldsymbol{\lambda}$ from (7.73) we first calculate $\nabla\omega$ from (7.87)

$$\nabla\omega = \frac{\mathbf{r} - ia\hat{\mathbf{k}}}{\omega} \tag{7.92}$$

where $\hat{\mathbf{k}}$ is the unit three-vector along the z axis, (0,0,1). It then follows by substitution into (7.73) that

$$\boldsymbol{\lambda} = \frac{2[\rho\mathbf{r} - a\sigma\hat{\mathbf{k}} + a(\mathbf{r} \times \hat{\mathbf{k}})]}{|\omega|^2 + r^2 + a^2} \tag{7.93}$$

We can simplify this by noting from (7.87) and (7.90) that

$$\begin{aligned} |\omega|^2 &= [(r^2 - a^2)^2 + 4a^2z^2]^{1/2} = 2\rho^2 - (r^2 - a^2) \\ |\omega|^2 + r^2 + a^2 &= 2(\rho^2 + a^2) \end{aligned} \tag{7.94}$$

Thus, $\boldsymbol{\lambda}$ can be written in quite simple form as

$$\boldsymbol{\lambda} = \frac{\rho}{\rho^2 + a^2}\left[\mathbf{r} + \frac{a^2z}{\rho^2}\hat{\mathbf{k}} + \frac{a}{\rho}(\mathbf{r} \times \hat{\mathbf{k}})\right] \tag{7.95}$$

or in terms of components

$$\begin{aligned} \lambda_1 &= \frac{\rho x + ay}{a^2 + \rho^2} \\ \lambda_2 &= \frac{\rho y - ax}{a^2 + \rho^2} \\ \lambda_3 &= \frac{z}{\rho} \end{aligned} \tag{7.96}$$

This may be compared with the Eddington form of the Schwarzschild solution as given in (7.83); for r much larger than a the two solutions are asymptotically equal, so that the present solution behaves like the Schwarzschild solution far from the source at the origin, as should be expected from the starting point (7.86).

Finally, let us put the pieces together as specified in (7.5) and (7.33) in order to display the Kerr line element

$$ds^2 = (dx^0)^2 - (d\mathbf{x})^2 - \frac{2m\rho}{\rho^4 + a^2z^2}\left[dx^0 + \frac{\rho}{a^2 + \rho^2}(x\,dx + y\,dy) + \frac{a}{a^2 + \rho^2}(y\,dx - x\,dy) + \frac{z}{\rho}dz\right]^2 \tag{7.97}$$

This is in the form obtained by Kerr in 1963.

7.6 Other Coordinates

The degenerate form of the metric tensor (7.5) has proved to be tremendously convenient for generalizing the Eddington form of the Schwarzschild solution to the Kerr solution, (7.97). We shall presently show that it actually represents the axially symmetric gravitational field of a rotating mass. Preparatory to showing this we wish to introduce a second set of coordinates that illustrate the axial symmetry very clearly. Then we shall introduce a third set of coordinates that is most convenient for analyzing the physical consequences.

As a radial coordinate we choose in preference to r the marker ρ as introduced in (7.87) or given explicitly in (7.90). It is then natural to introduce an angular coordinate θ as with the usual polar coordinates

$$\cos\theta = \frac{z}{\rho} \tag{7.98}$$

Next we introduce an angular coordinate φ by a convenient and elegant complex expression

$$(\rho - ia)e^{i\varphi}\sin\theta = x + iy \tag{7.99}$$

Finally we choose a time coordinate

$$u = x^0 + \rho \tag{7.100}$$

It is a simple task to express the line element (7.97) in terms of these new coordinates. The flat-space part is readily obtained from (7.98) and (7.99) by simple manipulations

$$dz^2 = (\cos\theta\, d\rho \mp \rho \sin\theta\, d\theta)^2 \tag{7.101}$$

$$\begin{aligned} dx^2 + dy^2 &= |d(x+iy)|^2 = |d(\rho - ia)e^{i\varphi}\sin\theta|^2 \\ &= (\sin\theta\, d\rho + a\sin\theta\, d\varphi + \rho\cos\theta\, d\theta)^2 \\ &\qquad + (\rho\sin\theta\, d\varphi - a\cos\theta\, d\theta)^2 \end{aligned}$$

Similarly, several of the other differential expressions that occur in the line element (7.97) are obtained in simple steps from (7.98) and (7.99) as

$$\begin{aligned} x\,dx + y\,dy &= \tfrac{1}{2}d|x+iy|^2 = \tfrac{1}{2}d[(\rho^2+a^2)\sin^2\theta] \\ &= (\rho^2+a^2)\sin\theta\cos\theta\, d\theta + \rho\sin^2\theta\, d\rho \end{aligned} \tag{7.102a}$$

$$\begin{aligned} x\,dy - y\,dx &= -\mathrm{Im}\,[(x+iy)d(x-iy)] \\ &= -\mathrm{Im}\,\{[(\rho - ia)e^{i\varphi}\sin\theta]d[(\rho+ia)e^{-i\varphi}\sin\theta]\} \\ &= (\rho^2+a^2)\sin^2\theta\, d\varphi + a\sin^2\theta\, d\rho \end{aligned} \tag{7.102b}$$

$$z\,dz = \rho\cos^2\theta\, d\rho - \rho^2\sin\theta\cos\theta\, d\theta \tag{7.102c}$$

Lastly, we note from (7.98) and (7.100)

$$\frac{2m\rho^3}{\rho^4 + a^2z^2} = \frac{2m\rho}{\rho^2 + a^2\cos^2\theta} \qquad dx^0 = du - d\rho \tag{7.103}$$

If the above expressions (7.101) and (7.103) are substituted into the line element (7.97), the result is a line element in the new coordinates given by

$$\begin{aligned} ds^2 &= \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right) du^2 - (\rho^2 + a^2\cos^2\theta)\, d\theta^2 \\ &\quad - \left[(\rho^2+a^2)\sin^2\theta + \frac{2m\rho a^2\sin^4\theta}{\rho^2 + a^2\cos^2\theta}\right] d\varphi^2 - 2du\, d\rho \\ &\quad - \frac{4m\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\, du\, d\varphi - 2a\sin^2\theta\, d\rho\, d\varphi \end{aligned} \tag{7.104}$$

There is no dependence of the line element on the angular coordinate φ, so that the solution (7.104) is manifestly axially symmetric.

A further simplification of this result can be made. We wish to show ultimately that the Kerr solution is appropriate to some form of rotating body, and so it is reasonable to suppose that space-time is, loosely speaking, dragged around with the body. This suggests that we attempt to make the line element formally as similar as possible to that of rotating flat space, (4.83). This form of the flat-space line element contains only one off-diagonal term in the metric tensor, a term in $d\varphi\, dt$. Motivated by this physical consideration, as well as mathematical simplicity, we shall introduce a coordinate transformation that eliminates the $du\, d\rho$ and $d\rho\, d\varphi$ terms of (7.104).

Let us write the line element (7.104) as

$$\begin{aligned} ds^2 &= g_{00}\, du^2 + g_{22}\, d\theta^2 + g_{33}\, d\varphi^2 + 2g_{03}\, du\, d\varphi \\ &\qquad + 2g_{01}\, du\, d\rho + 2g_{13}\, d\rho\, d\varphi \end{aligned} \tag{7.105}$$

We guess a simple form for the desired transformation

$$\begin{aligned} c\hat{t} &= u - A(\rho) \qquad & du &= c\, d\hat{t} + A'\, d\rho \\ \hat{\varphi} &= \varphi - B(\rho) \qquad & d\hat{\varphi} &= d\varphi + B'\, d\rho \end{aligned} \tag{7.106}$$

where $A(\rho)$ and $B(\rho)$ are functions only of ρ, to be determined, and a prime denotes differentiation with respect to ρ. If du and $d\varphi$ from (7.106) are substituted into (7.105), the result is

$$\begin{aligned} ds^2 &= g_{00}c^2\, dt^2 + (g_{00}A'^2 + g_{33}B'^2 + 2g_{01}A' + 2g_{13}B' \\ &\quad + 2g_{03}A'B')\, d\rho^2 + g_{22}\, d\theta^2 + g_{33}\, d\hat{\varphi}^2 + 2g_{03}c\, d\hat{t}\, d\hat{\varphi} \\ &\quad + 2(A'g_{03} + B'g_{33} + g_{13})\, d\hat{\varphi}\, d\rho + 2(A'g_{00} + B'g_{03} + g_{01})c\, d\hat{t}\, d\rho \end{aligned} \tag{7.107}$$

We now demand that the coefficients of $d\hat{\varphi}\, d\rho$ and $d\hat{t}\, d\rho$ be zero, which provides us with a simple set of linear equations for A' and B'. The solutions of these linear equations are

$$\begin{aligned} A' &= \frac{g_{33}g_{01} - g_{03}g_{13}}{g_{03}^2 - g_{00}g_{33}} = \frac{\rho^2 + a^2}{\rho^2 + a^2 - 2m\rho} \\ B' &= \frac{g_{00}g_{13} - g_{03}g_{01}}{g_{03}^2 - g_{00}g_{33}} = \frac{-a}{\rho^2 + a^2 - 2m\rho} \end{aligned} \tag{7.108}$$

It is very important to note that A' and B' are functions of ρ only. This allows one to integrate (7.108) for $A(\rho)$ and $B(\rho)$, although we shall not do this explicitly.

The line element is now given by (7.107) with the last two terms vanishing by construction. We can shorten the rather lengthy coefficient of $d\rho^2$ in (7.107).

$$(7.109)\quad g_{00}A'^2 + g_{33}B'^2 + 2g_{01}A' + 2g_{13}B' + 2g_{03}A'B' = A'(A'g_{00} + B'g_{03} + g_{01}) + B'(A'g_{03} + B'g_{33} + g_{13}) + g_{01}A' + g_{13}B' = g_{01}A' + g_{13}B'$$

In the last step we have utilized the fact that A' and B' are chosen so that the coefficients of $d\hat{\varphi}\, d\rho$ and $d\hat{t}\, d\rho$ in (7.107) are zero.

It is now straightforward to substitute the specific metric functions from (7.104) and (7.108) into the line element (7.107), with the coefficient of $d\rho^2$ given in (7.109)

$$(7.110)\quad ds^2 = \left(1 - \frac{2m\rho}{\rho^2 + a^2\cos^2\theta}\right)c^2\, d\hat{t}^2 - \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2 - 2m\rho}\, d\rho^2 - (\rho^2 + a^2\cos^2\theta)\, d\theta^2 - \left[(\rho^2 + a^2)\sin^2\theta + \frac{2m\rho a^2\sin^4\theta}{\rho^2 + a^2\cos^2\theta}\right] d\hat{\varphi}^2 - 2\,\frac{2m\rho a\sin^2\theta}{\rho^2 + a^2\cos^2\theta}\, c\, d\hat{t}\, d\hat{\varphi}$$

In this form (Boyer and Lindquist, 1967) the Kerr metric is manifestly axially symmetric, closely resembles the Schwarzschild solution in its standard form (6.53), and contains only one cross term, $d\hat{t}\, d\hat{\varphi}$, in analogy with the rotating flat-space line element (4.83).

Finally, it should be stressed once more that the marker ρ is quite as good a radial marker as the marker r previously used; in terms of either coordinate the Einstein equations are satisfied.

7.7 The Kerr Solution and Rotation

In this section we wish to demonstrate that the Kerr metric represents the field exterior to an axially symmetric rotating body, give a physical interpretation of the parameter a, and discuss some physical effects associated with the rotation of the source.

We first note that the Kerr metric in the form (7.110) has the general features expected for the field of a rotating source. (1) It is axially symmetric and time-independent. (2) It is unchanged if the signs of φ and t are both reversed. (We omit the hat in this section.) This is physically reasonable since φ and t are merely markers, and the signs are conventional: running time backward with a negative spin direction should be physically equivalent to running time forward with a positive spin direction. (3) For $a = 0$ it reduces to the Schwarzschild metric in the standard form (6.53). (4) It is unchanged if the signs of φ and a are both reversed, suggesting that a specifies a spin direction. (5) The presence of the $d\varphi\, dt$ cross term makes it superficially but suggestively similar to the metric of rotating flat space, (4.83).

The problem of physically interpreting the parameter a is made difficult by the lack of a classical analogue of the Kerr metric. In classical gravitational theory the field of an axially symmetric body is independent of its rotational motion, unlike the situation in relativity. Lacking a *direct* classical analogue, we shall instead use the results of an approximate relativistic calculation to build a bridge between classical and relativistic concepts. In 1918 Lense and Thirring studied the gravitational field of a spinning sphere of constant density. Using relativistic field equations valid in regions of space containing mass-energy (which we discuss in Chap. 10), they were able to obtain an approximate solution, valid for low rates of spin and weak fields, both inside and outside the sphere. The solution exterior to the sphere has the form (see Prob. 10.2)

$$(7.111)\quad ds^2 = \left(1 - \frac{2m}{r}\right)c^2\, dt^2 - \left(1 + \frac{2m}{r}\right)d\sigma^2 + 4\frac{\kappa J}{c^3 r}\sin^2\theta\, d\varphi\, c\, dt \qquad r = [x^2 + y^2 + z^2]^{1/2}$$

where m is the geometric mass, J is the angular momentum of the source, and $d\sigma^2$ is the flat-space line element in three dimensions. This solution is valid only to first order in the dimensionless quantities m/r and $\kappa J/c^3r^2$. This is a consistent expansion for rotating stars that are much larger than their Schwarzschild radius (see Exercise 7.7).

The Kerr solution (7.110) can be reduced to this approximate form in three steps. First we expand to first order in a/ρ:

$$(7.112)\quad ds^2 = \left(1 - \frac{2m}{\rho}\right)c^2\, dt^2 - \left(1 - \frac{2m}{\rho}\right)^{-1} d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - \frac{4ma}{\rho}\sin^2\theta\, d\varphi\, c\, dt$$

where to first order $\rho = r$. This is merely the exact Schwarzschild solution plus a cross term proportional to a. To compare with (7.111) we must put this into isotropic form; this is very easily done using the same change of radial coordinate as we used in Chap. 6 to put the Schwarzschild line element into isotropic form, $\rho = \hat{\rho}(1 + m/2\hat{\rho})^2$, where $\hat{\rho}$ is the isotropic radial marker. This leads to (the algebra need not be repeated; see Sec. 6.2)

$$(7.113)\quad ds^2 = \frac{(1 - m/2\hat{\rho})^2}{(1 + m/2\hat{\rho})^2}\, c^2\, dt^2 - \left(1 + \frac{m}{2\hat{\rho}}\right)^4 d\sigma^2 - \frac{4ma}{\hat{\rho}(1 + m/2\hat{\rho})^2} \sin^2\theta\, d\varphi\, c\, dt$$

Finally we expand to first order in $m/\hat{\rho}$

$$(7.114)\quad ds^2 = \left(1 - \frac{2m}{\hat{\rho}}\right) c^2\, dt^2 - \left(1 + \frac{2m}{\hat{\rho}}\right) d\sigma^2 - \frac{4ma}{\hat{\rho}} \sin^2\theta\, d\varphi\, c\, dt$$

which agrees with (7.111) if we identify

$$(7.115)\quad ma = -\frac{\kappa J}{c^3}$$

Thus a is a measure of the angular momentum per unit mass of the source. We shall refer to ma as the geometric angular momentum in the same spirit as we refer to m as the geometric mass. The negative sign in (7.115) should be stressed; a body rotating in a positive sense will have a positive J and a negative a.

This correspondence argument has relied on an approximate solution for a spherical rotating source. However, we need not conclude that the exact Kerr solution necessarily corresponds to a spherical source and will use only the fact that the source has a geometric angular momentum equal to ma without reference to its structure (see Prob. 11.3).

A very interesting physical effect results from the rotational nature of the Kerr solution; a body in geodesic motion experiences a force proportional to the parameter a reminiscent of a Coriolis force. Loosely speaking, we may think of the rotating source as "dragging" space around with it; in a Machian sense the source "competes" with the Lorentzian boundary conditions at infinity in the establishment of a local inertial frame. To demonstrate this force we proceed to obtain the geodesic equations of motion, precisely as in Chap. 6 [see (6.74) to (6.78)]. For simplicity we shall consider the approximate form (7.111) and work only to first order

in a/ρ. The equations of motion are

$$\frac{d}{ds}(\rho^2\dot{\theta}) = \rho^2 \sin\theta\cos\theta\,\dot{\varphi}^2 + \frac{4ma}{\rho}\sin\theta\cos\theta\, c\dot{t}\dot{\varphi}$$

$$\frac{d}{ds}\left(\rho^2\sin^2\theta\,\dot{\varphi} + \frac{2ma}{\rho}\sin^2\theta\, c\dot{t}\right) = 0$$

$$(7.116)\quad \frac{d}{ds}\left(\left(1 - \frac{2m}{\rho}\right)c\dot{t} - \frac{2ma}{\rho}\sin^2\theta\,\dot{\varphi}\right) = 0$$

$$1 = \left(1 - \frac{2m}{\rho}\right)c^2\dot{t}^2 - \left(1 - \frac{2m}{\rho}\right)^{-1}\dot{\rho}^2 - \rho^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) - \frac{4ma}{\rho}\sin^2\theta\, c\dot{t}\dot{\varphi}$$

We limit ourselves to the special case of equatorial orbits, $\theta = \pi/2$, which is particularly simple (note, however, that the general orbit in the Kerr metric does *not* lie in a plane, unlike the Schwarzschild case):

$$(7.117a)\quad \frac{d}{ds}\left(\rho^2\dot{\varphi} + \frac{2ma}{\rho}c\dot{t}\right) = 0$$

$$(7.117b)\quad \frac{d}{ds}\left[\left(1 - \frac{2m}{\rho}\right)c\dot{t} - \frac{2ma}{\rho}\dot{\varphi}\right] = 0$$

$$(7.117c)\quad 1 = \left(1 - \frac{2m}{\rho}\right)c^2\dot{t}^2 - \left(1 - \frac{2m}{\rho}\right)^{-1}\dot{\rho}^2 - \rho^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) - \frac{4ma}{\rho}\sin^2\theta\, c\dot{t}\,\dot{\varphi}$$

The conserved angular momentum conjugate to φ is not $\rho^2\dot{\varphi}$, as in the Schwarzschild problem, but the quantity indicated in (7.117a); an extra term proportional to a appears.

We now consider, as in Sec. 4.2, a body in instantaneous radial motion, $\dot{\varphi} = 0$. Equation (7.117a) then yields an equation for $\ddot{\varphi}$

$$(7.118)\quad \rho\ddot{\varphi} + \frac{2ma}{\rho^2}c\ddot{t} - \frac{2ma}{\rho^3}c\dot{t}\dot{\rho} = 0$$

while (7.117b) gives

$$(7.119)\quad c\dot{t} = l\left(1 - \frac{2m}{\rho}\right)^{-1} \qquad c\ddot{t} = \frac{2ma}{\rho(1 - 2m/\rho)}\ddot{\varphi} - \frac{2m\dot{\rho}l}{\rho^2(1 - 2m/\rho)^2}$$

We substitute these back into (7.118) and obtain, to first order in a/ρ,

$$\rho\ddot{\varphi} + 2\left[\frac{-ma}{\rho^3(1-2m/\rho)}\right]\left[1 + \frac{4m}{\rho(1-2m/\rho)}\right]\dot{\rho} = 0 \tag{7.120}$$

and to first order in m/ρ

$$\rho\ddot{\varphi} + 2\left(\frac{-ma}{\rho^3}\right)\dot{\rho} = 0 \tag{7.120$'$}$$

This is identical in form to the Coriolis equation (4.123) with a function of ρ replacing the constant angular velocity w. This Coriolislike force is of course fundamentally different from the true Coriolis force in (4.123), which can be transformed away globally by a coordinate transformation.

7.8 Distinguished Surfaces and the Rotating Black Hole

In the Kerr solution two surfaces arise which are analogous to the Schwarzschild singular surface and which are of great physical interest, e.g., in the gravitational collapse of a rotating star. We devote this section to their study.

Let us first study the red shift of light emitted from a source at rest in the Kerr geometry. We obtained a general solution to this problem in Eq. (4.152):

$$\nu = \nu_0\left(\frac{g_{00}(x_s{}^\mu)}{g_{00}(x^\mu)}\right)^{1/2} \tag{7.121}$$

Here ν is the frequency of the light observed at x^μ, and ν_0 is the proper frequency of the light emitted by the source at rest at $x_s{}^\mu$. [Note that the last term appearing in Eq. (4.152) is an approximation and not appropriate to the present discussion.] It is clear from (7.110) that for large values of ρ the red shift in the Kerr metric is approximately equal to that in the Schwarzschild metric, since the g_{00} functions differ only by terms of second order in a/ρ. However, the Kerr metric displays two surfaces of infinite red shift, both of which are intrinsically different from the Schwarzschild surface of infinite red shift. These occur where $g_{00}(x_s{}^\mu) = 0$, which from (7.121) makes $\nu = 0$. Setting g_{00} in (7.110) equal to zero, we obtain two axially symmetric surfaces

$$\rho = m \pm \sqrt{m^2 - a^2\cos^2\theta} \tag{7.122}$$

In the limit of $a \to 0$ these surfaces reduce to the Schwarzschild surface $\rho = 2m$, for the plus sign, and the origin $\rho = 0$, for the minus sign. The surface corresponding to the plus sign is of much greater physical interest; it is an axially symmetric surface with a radius at the equator of $2m$ and a radius at the poles of $m + \sqrt{m^2 - a^2}$. We shall assume that $|a| < m$ so that the surface is well defined; that is, ρ is real. The surface corresponding to the minus sign is completely contained within the above surface, $\rho_\infty \equiv m + \sqrt{m^2 - a^2\cos^2\theta}$.

Let us note that the outer infinite red shift surface is comparable in radius to the Schwarzschild surface, so that for ordinary stars like the sun it is of little physical consequence, lying well *inside* the star, where the vacuum field equations are not valid. Only for extremely dense stars can regions near this surface be in free space.

The Schwarzschild singular surface studied in Chap. 6 plays two unusual roles: (1) it serves as an infinite red shift surface for sources at rest, and (2) it acts as a one-way membrane for physical objects whose trajectories lie in or on the forward light cone, as discussed in Sec. 6.8. In the Kerr metric these roles are played by different surfaces. To find the one-way membranes of the Kerr solution we shall introduce and study the concept of a so-called null hypersurface, i.e., a hypersurface whose normal vector is null. Such a null hypersurface always acts as a one-way membrane, as we shall show.

Consider a smooth hypersurface S defined by the equation

$$u(x^\mu) = \text{const} \tag{7.123}$$

the vector $n_\alpha = u_{|\alpha}$ is a normal to S since its inner product with any dx^α contained in S is zero:

$$n_\alpha\, dx^\alpha = u_{|\alpha}\, dx^\alpha = du = 0 \tag{7.124}$$

At any point P on S we introduce a locally Lorentzian metric so that the line element is

$$ds^2 = (dx^0)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{7.125}$$

and the local light cone, defined as the local hypersurface $ds^2 = 0$, is very simple. Moreover, by a rotation in three-space we can place the three-vector part of n^α along the x axis

$$n^\alpha = (n^0, n^1, 0, 0) \qquad n^\alpha n_\alpha = (n^0)^2 - (n^1)^2 \tag{7.126}$$

This form of the normal vector restricts the form of any vector t^α tangent to S at P since the two must be orthogonal:

$$n_\alpha t^\alpha = n^0 t^0 - n^1 t^1 = 0 \qquad \frac{t^0}{t^1} = \frac{n^1}{n^0} \tag{7.127}$$

thus t^α must have the form

$$t^\alpha = \lambda(n^1, n^0, a, b) \tag{7.128}$$

where λ, a, and b are arbitrary. It follows that the norm of t^α is

$$\begin{aligned} t^\alpha t_\alpha &= \lambda^2[(n^1)^2 - (n^0)^2 - a^2 - b^2] \\ &= -\lambda^2(n^\alpha n_\alpha + (a^2 + b^2)) \end{aligned} \tag{7.129}$$

This simple relation between the norms of the normal and tangent vectors of S leads to a beautiful geometrical result. We must consider three cases in the light of (7.129).

Case I: n^α is timelike, $n^\alpha n_\alpha > 0$. Then $t^\alpha t_\alpha$ is negative-definite so that t^α is spacelike and *no* tangent vector to S can lie on the local light cone, i.e., be null.

Case II: n^α is null, $n^\alpha n_\alpha = 0$. Then $t^\alpha t_\alpha$ is negative *except* when $a = b = 0$, in which case it is zero. Thus there is *one* unit tangent vector to S, which, along with its multiples, lies on the local light cone at P.

Case III: n^α is spacelike, $n^\alpha n_\alpha < 0$. Then $t^\alpha t_\alpha$ may have either sign or be zero. In particular $t^\alpha t_\alpha = 0$ on the circle defined by $a^2 + b^2 = -n^\alpha n_\alpha > 0$. Thus there is a *whole family* of vectors tangent to S, which also lie on the local light cone.

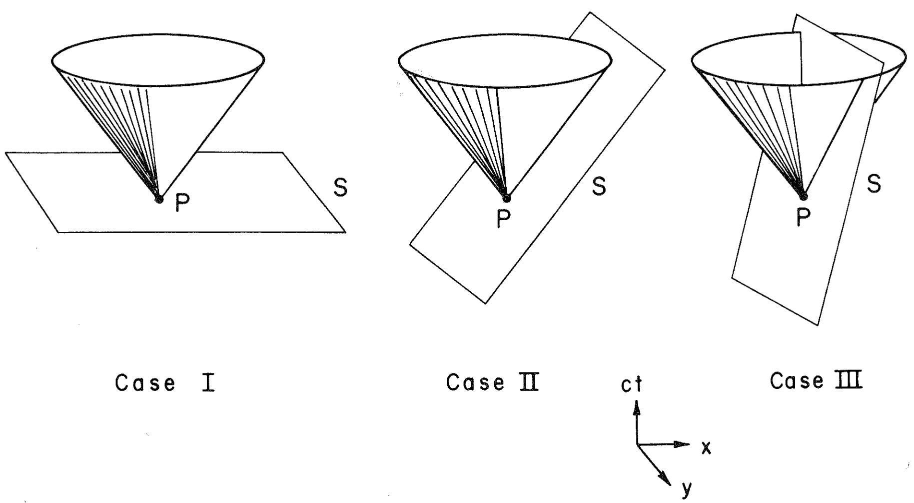


Fig. 7.1
The three possible relations of a hypersurface S to the local light cone, with one space dimension suppressed.

We may picture the above results if we suppress the z coordinate and consider only two spatial coordinates and one time coordinate. The geometry is shown in Fig. 7.1. It is clear also that the final statement of the result has an invariant meaning.

The physical interpretation of this result is easy. Since light has a trajectory on the forward light cone and massive bodies have trajectories within the forward light cone, we see that physical objects can pass through a spacelike hypersurface in *either* direction and can pass through a timelike hypersurface in *only one* direction. The null hypersurface is the critical case: it is the configuration where the one-way behavior begins, and we may therefore identify it as a *one-way membrane*. A very simple example of this behavior can be found in special relativity. The hypersurface t = const is timelike, and physical objects can pass it in only one direction; x = const is spacelike, and physical objects may pass in either direction; while $ct - x = 0$ is null and acts as a one-way membrane. In the last case, a tangent vector of the null hypersurface that lies along the local light cone is $t^\alpha = (1,1,0,0)$. A second example of a one-way membrane is provided by the Schwarzschild singular surface. A spherical surface r = const in the Schwarzschild geometry has a normal vector

$$n_\alpha = (0,1,0,0) \qquad n^\alpha n_\alpha = -\left(1 - \frac{2m}{r}\right) \tag{7.130}$$

Thus as r decreases through $2m$ the spherical surface changes from spacelike to null to timelike. Unlike the above example in special relativity, this surface has the important feature that it is finite in spatial extent.

In Chap. 8 we shall discuss null hypersurfaces further. They occur also as characteristic hypersurfaces of the Einstein equations, the hypersurfaces over which the second derivatives of the metric may be discontinuous, analogous to the characteristic surfaces of Maxwell's equations discussed in Chap. 4.

We now search for the null hypersurfaces of the Kerr geometry. It is easy to check that the outer infinite red shift surface (7.122) is not a null hypersurface; the normal vector and its norm are

$$(7.131)\qquad \begin{aligned} n_\alpha &= \left(0,\, 1,\, -\frac{a^2 \cos\theta \sin\theta}{\sqrt{m^2 - a^2\cos^2\theta}},\, 0\right) \\ n^\alpha n_\alpha &= -\frac{\rho^2 + a^2 - 2m\rho + (a^4\cos^2\theta\sin^2\theta)/(m^2 - a^2\cos^2\theta)}{\rho^2 + a^2\cos^2\theta} \end{aligned}$$

Since this is clearly negative, the infinite red shift surface will pass physical objects in both directions and is not a one-way membrane. We shall seek an axially symmetric and time-dependent null hypersurface

$$(7.132)\qquad u(r,\theta) = \text{const} \qquad n_\alpha = \left(0, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, 0\right)$$

A differential equation for u results from setting the norm of n_α equal to zero

$$(7.133)\qquad (\rho^2 - 2m\rho + a^2)\left(\frac{\partial u}{\partial \rho}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2 = 0$$

This is separable and easily solved; we set up a product solution

$$(7.134)\qquad u(\rho,\theta) = R(\rho)\Theta(\theta)$$

and find

$$(7.135)\qquad -(\rho^2 - 2m\rho + a^2)\left(\frac{\partial R/\partial\rho}{R}\right)^2 = \left(\frac{\partial\Theta/\partial\theta}{\Theta}\right)^2$$

Since the left side of this equation is a function of ρ alone and the right side a function of θ alone, both must be equal to a positive constant, which we may call λ. Thus

$$(7.136)\qquad \frac{\partial\Theta}{\partial\theta} = \sqrt{\lambda}\,\Theta \qquad \Theta = A\exp(\sqrt{\lambda}\,\theta)$$

where A is an arbitrary constant. However, this is not an acceptable solution in general since it is not periodic in θ and therefore does not correspond to a real surface unless $\lambda = 0$, which implies $\Theta = $ const. With $\lambda = 0$ we then obtain

$$(7.137)\qquad \left(\frac{\partial R/\partial\rho}{R}\right)^2 (\rho^2 - 2m\rho + a^2) = 0$$

The solution $\partial R/\partial\rho = 0$ may be rejected, and we are being left with

the two solutions

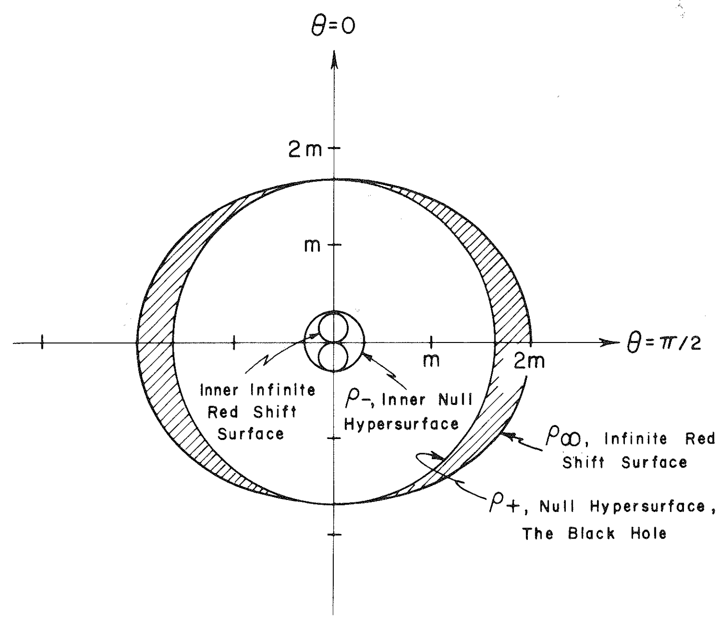
$$(7.138)\qquad \rho_\pm = m \pm \sqrt{m^2 - a^2}$$

It is remarkable that we obtain spherical surfaces and that these are well defined only so long as $|a| < m$. In the limit of $a \to 0$ these two surfaces reduce to the Schwarzschild surface $\rho = 2m$, and the origin $\rho = 0$.

The outer one-way membrane, $\rho_+ = m + \sqrt{m^2 - a^2}$, represents a dividing surface between the region of the Kerr geometry that is accessible from the distant exterior (ρ much greater than m and a) and that which is not. It is for this reason that the inner infinite red shift surface, $\rho_- = m - \sqrt{m^2 - a^2\cos^2\theta}$, is not of great physical significance; it is entirely contained within the outer one-way membrane. On the other hand, the outer infinite red shift surface is exterior to the one-way membrane, and the region between has several interesting features. Our further discussion will deal only with the *outer* one-way membrane and the outer infinite red shift surface (see Fig. 7.2).

We have shown that the infinite red shift surface does not represent a barrier to either massive test bodies or light; both may cross the surface in either direction, except at the special points $\theta = 0$ and π, where the

Fig. 7.2
The distinguished surfaces for the Kerr metric; $a = m/2$ in this illustration.



infinite red shift surface and the one-way membrane coincide. It is interesting to consider the physical significance of this for a star whose surface has approached the infinite red shift surface, $\rho = \rho_\infty$. For the special case of a spherically symmetric nonrotating star whose exterior field is described by the Schwarzschild metric we expect that atoms on the stellar surface will be at coordinate rest, that is, $dr = d\theta = d\varphi = 0$ along their world-line. For the case of a rotating star, however, this is not true; atoms on the stellar surface will in general have a velocity in the φ direction. Thus their world-lines in the Kerr metric will not correspond to coordinate rest. Since the infinite red shift surface refers explicitly to sources at rest in the Kerr metric, we may not conclude that a star whose surface approaches the infinite red shift surface will become a black hole analogous to the Schwarzschild black hole; light may escape from the surface, dependent upon its actual motion.

To further emphasize that coordinate rest is not a reasonable state for an atom in the Kerr metric let us look further into the physical interpretation of the world-lines $dr = d\theta = d\varphi = 0$. We have $ds^2 = g_{00}c^2\,dt^2$ for such a world-line. Clearly ds^2 is positive outside, zero on, and negative inside the infinite red shift surface, since it has the sign of g_{00}. All massive bodies have world-lines for which $ds^2 > 0$, however, so that coordinate rest is possible only for an atom outside the infinite red shift surface. Curiously a body at *coordinate rest* on the surface moves with *physical velocity* c since $ds^2 = 0$. (Equivalently its velocity is c in a tangent Lorentz space.)

The nature of tne one-way membrane $\rho = \rho_+$ is very different from the infinite red shift surface; it represents a surface from which no light ray may emanate, regardless of the motion of the source. The radius ρ_+ may thus be regarded as the true critical size for which a rotating star becomes a black hole, analogous to the Schwarzschild black hole.

7.9 Effective Potentials and Black Hole Energetics

In classical mechanics the central-force problem can be treated as a one-dimensional problem by the introduction of an effective potential energy, which combines the usual potential energy and a "centrifugal potential energy." In this section we shall show that an analogue of the classical effective-potential-energy function can be constructed for test bodies moving in both the Schwarzschild and Kerr metrics. We shall treat the Schwarzschild metric first because of its simplicity and similarity to the classical problem; the Kerr metric is fundamentally different and possesses very interesting energetic properties which may be of great interest in astrophysical problems.

We begin with (6.82) describing a test body of rest mass μ and give a physical interpretation of the constant l. If the motion is unbounded so that the body may escape, we have asymptotically for large r

$$\dot{r}^2 = c^2l^2 - 1 \tag{7.139}$$

Moreover, we may express $\dot{r}$, using (6.81), as

$$\dot{r} = \frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = \frac{dr}{dt}\,l\left(1 - \frac{2m}{r}\right)^{-1} \to vl \tag{7.140}$$

The limit $dr/dt \to v$ follows since only the r component of the velocity remains finite asymptotically, as is evident from (6.80). We may now combine (7.139) and (7.140) to obtain l in terms of the asymptotic velocity

$$cl = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \equiv \gamma \tag{7.141}$$

where the sign of l is chosen positive for obvious convenience. This quantity γ is precisely the Lorentz contraction factor. A useful equivalent form of (7.141) follows if we observe that in the asymptotic flat space where special relativity holds the total energy E of the body is given by its rest energy μc^2 times γ. Thus we may express the constant cl as

$$cl = \frac{E}{\mu c^2} \tag{7.142}$$

where E is commonly termed the energy at infinity; it is a constant of the motion, is equal to the energy at infinity, and can therefore be defined as the general relativistic generalization of the total energy.

To define the effective potential energy function we solve (6.82) for $\mu^2\dot{r}^2$

$$\begin{aligned}\mu^2\dot{r}^2 &= \mu^2l^2c^2 - \left(1 - \frac{2m}{r}\right)\left(\frac{\mu^2h^2}{r^2} + \mu^2\right) \\ &= \frac{E^2}{c^4} - \left(1 - \frac{2m}{r}\right)\left(\frac{\mu^2h^2}{r^2} + \mu^2\right) \equiv \frac{E^2}{c^4} - \frac{V^2}{c^4}\end{aligned} \tag{7.143}$$

The function V introduced here will be referred to as the effective potential energy; the positive root is always used. We see that the difference between the total energy and the effective potential energy provides a measure of the radial velocity of the particle.

One immediate use of the effective potential energy is in relating turning points to the constants E and h. A turning point is a point where $\dot{r} = 0$ and is thus obtained as a solution of $E^2 - V^2 = 0$.

The physical interpretation of V is most clearly seen by taking the classical limit of low velocities and weak fields. We may express V up to order mass$^2/r^2$ by expanding (7.143), using the positive root:

$$V \cong \mu c^2 - \frac{\kappa M \mu}{r} + \frac{\mu^2 c^2 h^2}{2\mu r^2} \qquad \frac{\kappa M}{c^2} \equiv m \tag{7.144}$$

From (6.80) it is evident that in the classical limit $\mu c h$ is the usual angular momentum L. Thus (7.144) states that V is approximately μc^2 plus the classical effective potential energy

$$\begin{aligned} V &\cong \mu c^2 + V_{\text{cl}} \\ V_{\text{cl}} &\equiv -\frac{\kappa M \mu}{r} + \frac{L^2}{2\mu r^2} \end{aligned} \tag{7.145}$$

If we now write E as $\mu c^2 + E_{\text{cl}}$, where $E_{\text{cl}} \ll \mu c^2$ is the usual classical total energy, we can write Eq. (7.143) as

$$\begin{aligned} \mu^2 \dot{r}^2 &\cong \frac{(\mu c^2 + E_{\text{cl}})^2}{c^4} - \frac{\mu c^2 + V_{\text{cl}}}{c^4} \\ &\cong \frac{2\mu E_{\text{cl}}}{c^2} - \frac{2\mu V_{\text{cl}}}{c^2} \end{aligned} \tag{7.146}$$

or

$$E_{\text{cl}} \cong \frac{\mu(\dot{r}c)^2}{2} + V_{\text{cl}} \cong \frac{\mu}{2}\left(\frac{dr}{dt}\right)^2 + V_{\text{cl}} \tag{7.147}$$

where we have neglected small terms of second order. This is the usual classical expression involving the effective potential energy, thus justifying our statement that V defined in (7.143) is a relativistic generalization of this quantity. For cases of bounded motion we may still define E as the total energy, using the above classical reduction as motivation.

The effective-potential-energy function is useful in analyzing the motion of bodies in the Schwarzschild field as illustrated in Exercise 7.9. We shall next define an analogous function for the Kerr metric which is of fundamental importance. For simplicity we shall limit our discussion to the special case of equatorial orbits. Our discussion relies on and follows very closely that of the Schwarzschild case.

The exact equations of motion for the Kerr metric (7.110) are easily obtained, as in the Schwarzschild metric. We might expect equatorial orbits to exist due to the symmetry of the Kerr metric about the equatorial plane. Indeed if we set $\theta = \pi/2$ in the equations of motion, as we did for the Schwarzschild problem in Sec. 6.3, we obtain the following consistent set of equations for equatorial orbits:

$$\left(1 - \frac{2m}{\rho}\right) c\dot{t} - \frac{2ma}{\rho}\dot{\varphi} = cl \tag{7.148a}$$

$$\left[\rho^2 + a^2\left(1 + \frac{2m}{\rho}\right)\right]\dot{\varphi} + \frac{2ma}{\rho} c\dot{t} = h \tag{7.148b}$$

$$\begin{aligned} \left(1 - \frac{2m}{\rho}\right) c^2\dot{t}^2 - \left(1 - \frac{2m}{\rho} + \frac{a^2}{\rho^2}\right)^{-1} \dot{\rho}^2 \\ - \left[\rho^2 + a^2\left(1 + \frac{2m}{\rho}\right)\right]\dot{\varphi}^2 - \frac{4ma}{\rho} c\dot{t}\dot{\varphi} = 1 \end{aligned} \tag{7.148c}$$

We can identify cl, precisely as before, as the total energy, or energy at infinity, of the test body divided by μc^2, where μ is the test-body mass; this is evident by inspection of (7.148) for very large ρ since in this limit all terms in a vanish and the preceding discussion applies unchanged.

Equations (7.148a) and (7.148b) may be easily solved for $\dot{\varphi}$ and $\dot{t}$

$$\begin{aligned} Dc\dot{t} &= \left[\rho^2 + a^2\left(1 + \frac{2m}{\rho}\right)\right] cl + \frac{2ma}{\rho} h \\ D\dot{\varphi} &= -\frac{2ma}{\rho} cl + \left(1 - \frac{2m}{\rho}\right) h \\ D &\equiv a^2 + \rho^2\left(1 - \frac{2m}{\rho}\right) \end{aligned} \tag{7.149}$$

The above expressions may be substituted into (7.148c) and an equation for $\mu^2\dot{\rho}^2$ obtained; after elementary manipulation the result is

$$\begin{aligned} \mu^2\dot{\rho}^2 = \frac{1}{\rho^3}\Bigg\{\frac{[\rho^3 + a^2(\rho + 2m)]E^2}{c^4} + \frac{(2m - \rho)L^2}{c^2} \\ + \frac{4maLE}{c^3} - \mu^2\rho^2(\rho - 2m) - a^2\rho\mu^2\Bigg\} \end{aligned} \tag{7.150}$$

where, as before, we define $L = \mu h c$. This equation is the generalization of (7.143), which is used to define the Schwarzschild effective potential;

for $a = 0$ they are identical. In the present case a cross term occurs, and we may define two effective potentials by writing (7.150) as

$$\mu^2\dot{\rho}^2 = \frac{1}{c^4}(E - V_+)(E - V_-) \tag{7.151}$$

where $V_\pm$ are the roots of the polynomial in E that constitutes the right side of (7.150). These are explicitly

$$V_\pm = \frac{-2maLc \pm [\rho^2 + a^2 - 2m\rho]^{1/2}[c^2\rho^2L^2 + \mu^2\rho c^4(\rho^3 + a^2\rho + 2ma)]^{1/2}}{\rho^3 + a^2(\rho + 2m)} \tag{7.152}$$

For large ρ, $V_\pm \to \pm\mu$; thus to generalize the classical results and the Schwarzschild effective potential we choose the positive sign and identify V_+ as the physically significant effective potential. In the limit of small a we readily obtain

$$V_+ \cong V_s + \frac{2maLc}{\rho^3} \tag{7.153}$$

where V_s is the Schwarzschild effective potential. There is no direct classical analogue of the extra term on the right side of (7.153).

The Schwarzschild effective potential is positive definite for $r > 2m$. It is extraordinary that the Kerr effective potential does not possess this feature. Indeed, at the black-hole radius, where $\rho^2 + a^2 - 2m\rho = 0$, V_+ is given by

$$V_+ = \frac{-2maLc}{\rho^3 + a^2(\rho + 2m)} \tag{7.154}$$

which will be negative if a and L have the same sign. Since a has the opposite sign of the black-hole angular momentum J by (7.115), this means that the test body has the opposite sign of angular momentum L, that is, is counterrotating. This may be interpreted physically as follows: a body released with $\dot{\rho} = 0$ at ρ will have total energy $E = V_+(\rho)$. This total energy can be negative for the Kerr metric, which corresponds to a binding energy in excess of the rest energy and leads to the possibility of interesting physical effects, as we shall presently discuss. Let us return to (7.152) and ask for what values of ρ the effective potential V_+ can be negative. Clearly the mass term μ^2 contributes positively to V_+, so to seek minimum V_+ we consider the limit of $\mu \to 0$. Then V_+

will be zero or negative when

$$(\rho^2 - 2m\rho + a^2)\rho^2L^2 \leq 4m^2a^2L^2 \tag{7.155}$$

The maximum solution for zero energy is $\rho = 2m$. We therefore see that a test body of zero mass can have a negative total energy for ρ between $m + \sqrt{m^2 - a^2}$ and $2m$.

The above analysis may be generalized to nonequatorial orbits, in which case one would find that negative total energy is possible between the null or black-hole surface $\rho = m + \sqrt{m^2 - a^2}$ and the infinite red shift surface $\rho = m + \sqrt{m^2 - a^2\cos^2\theta}$. This region is commonly called the *dynamic zone* or *ergosphere*. The infinite red shift surface, which previously had little physical significance, is now seen to play an important role in the energetics of the black hole.

By using orbits entering the ergosphere, energy can be extracted from a Kerr black hole (Penrose, 1969). One sends a particle into the ergosphere with energy E_1 and lets it decay into two particles with energies E and E_2 in such a way that E is negative, as we have shown is possible. Then overall conservation of energy implies that E_2 will be larger than E_1. The rotational energy of the black hole must in general decrease in the process. This raises the possibility that rotating black holes may be sources of large amounts of energy. If the energy can be extracted in the form of electromagnetic or gravitational radiation, this would be of great interest in the study of quasars, whose energy source remains a mystery.

Exercises

7.1 Explicitly verify the general properties of a degenerate metric (7.8), (7.9), and $g = -1$ for Eddington's special case (7.4).

7.2 Show that the order m^4 equations (7.15*d*) are satisfied identically, and verify also Eq. (7.16).

7.3 In the text $l_0^2 = \alpha$ is shown to be a solution of (7.25). To show that it is unique write the general solution as $l_0^2 = f\alpha$, with f an arbitrary function, and show by steps analogous to those following (7.65) that $\nabla f = 0$. Thus f must be a constant, and l_0^2 is unique up to this multiplicative constant.

7.4 Prove that

$$A = \frac{\beta^2 - \alpha^2}{2l_0} \quad \text{and} \quad L = \frac{\beta^2 + 3\alpha^2}{2l_0}$$

7.5 Show that no loss of generality occurs if the displacement discussed in Sec. 7.5 is assumed to have the form (7.86); do this by beginning with general complex a_i in (7.85) and then performing real physical rotations and translations to obtain (7.86).

7.6 From (7.98) and (7.99) compute $\cos^2\theta + \sin^2\theta$ and use (7.89) to show that it is 1, thereby showing that θ is a reasonable angular coordinate.

7.7 The dimensionless quantities $\alpha_1 = m/r$ and $\alpha_2 = \kappa J/c^3r^2$ occur in the Lense-Thirring metric as expansion coefficients. Show that for a reasonable star the ratio $\alpha_2/\alpha_1 < 1$ outside the surface. Show also that for a star of reasonable mass and rapid rotation one may have $\alpha_1^2/\alpha_2 \ll 1$. Thus there exist situations where the first-order approximations of the Lense-Thirring metric are justified.

7.8 Discuss the positions of the singularities of γ for the Kerr and Schwarzschild metrics. How do the singularities relate to the symmetry of the fields?

7.9 Use the Schwarzschild potential-energy function to show that the minimum radius for stable circular orbits is $r = 6m$.

7.10 In general the Kerr metric allows no radial null geodesics unless $a = 0$; show this, and consider the exceptional case of a null geodesic along the z axis of axial symmetry. Then consider null geodesics which are radial for large r and study their behavior as a function of r, that is, how do they deviate in the φ direction for small r? What can be said about the light-trapping properties of a rotating black hole versus a nonrotating black hole?

7.11 Consider a metric of the form

$$ds^2 = f(r)c^2\,dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

and show that it can be put into the degenerate form considered in this chapter by a coordinate transformation involving only the time.

Problems

7.1 Show, using the degenerate metric form (7.5), that

$$R^\alpha{}_{\beta\gamma\delta}l^\beta l^\delta = -2mA^2 l_\gamma l^\alpha$$

and thus that

$$R_{\alpha\beta\delta\gamma}l_\omega l^\beta l^\delta - R_{\alpha\beta\delta\omega}l_\gamma l^\beta l^\delta = 0$$

Penrose (1960) has shown that this equation implies that the space-time is algebraically special, i.e., it is not of Petrov type I.

7.2 In the study of geometrical optics in a Riemann space, quantities called the *expansion*, the *twist*, and the *shear* of a family of light rays characterize the shadow cast by a two-dimensional disk on a nearby screen. Study these quantities (Pirani, 1965) and show that for the present situation the expansion is $-\alpha$, the twist is β, and the shear is zero.

7.3 As noted in the text, coordinate rest is not a possible condition for a physical object inside the infinite red shift surface. Show that at any given point there exists a coordinate transformation from t and φ to $\hat{t}$ and $\hat{\varphi}$ such that coordinate rest in terms of $\hat{t}$ is possible for a physical object. Thus show that t is not a good time marker inside the infinite red shift surface but $\hat{t}$ is (see Vishveshwara, 1968).

7.4 Show that there exist physically acceptable trajectories which cross the infinite red shift and continue to spatial infinity (see Carter, 1968).

7.5 The Penrose process discussed in Sec. 7.9 is potentially capable of extracting large amounts of energy from a rotating black hole. Study this process further and discuss the mechanism and efficiency of extraction (Penrose, 1969; Christodolou, 1970). In particular show that energy of the order of one-third the total mass energy may be extracted from a rapidly spinning black hole.

7.6 We studied equatorial orbits in the text to understand the energy properties of the dynamic zone, or ergosphere. Study the work of Carter (1968), in particular the implications of the so-called Killing tensor, and generalize the discussion to show that there exist negative energy orbits in the entire ergosphere.

7.7 The metric form considered in this chapter corresponds to an algebraically special space-time, as noted in Exercise 7.11 and Prob. 7.1. Show that not all algebraically special space-times have a metric of this form.

7.8 What is the Petrov type of the Kerr solution? (See also Prob. 7.1.)

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CHAPTER

8

The Mathematical Structure of the Einstein Differential System; the Problem of Cauchy

Einstein's equations in a matter-free region form a system of 10 second-order quasi-linear differential equations in the four space-time variables. These equations are to be solved for the 10 unknown components of the metric tensor $g_{\alpha\beta}$, which we may interpret as gravitational potentials. The equations are

$$R_{\mu\nu} = 0 \tag{8.1}$$

or equivalently, by the definition (5.119),

$$\left\{ {\sigma \atop \mu\ \sigma} \right\}_{|\nu} - \left\{ {\sigma \atop \mu\ \nu} \right\}_{|\sigma} + \left\{ {\alpha \atop \mu\ \sigma} \right\} \left\{ {\sigma \atop \alpha\ \nu} \right\} - \left\{ {\alpha \atop \mu\ \nu} \right\} \left\{ {\sigma \atop \alpha\ \sigma} \right\} = 0 \tag{8.2}$$

In this form the quasi-linear nature of the equations is apparent; only the two first terms contain the second derivatives of the $g_{\alpha\beta}$'s, and they are linear in them, whereas they and the other terms are obviously not linear in the first derivatives or in the $g_{\alpha\beta}$'s themselves.

Equations (8.2) connect the time derivatives of the components of the metric tensor $g_{\mu\nu}$ with the space derivatives and the $g_{\mu\nu}$ themselves. They allow the solution of the following mathematical problem: Given the metric tensor and all its first derivatives at a given moment x^0 in the entire three-dimensional space of the remaining three variables x^i, to compute its value for all future time. This is a typical initial-value problem in the theory of partial differential equations and represents the causal development of a physical system from initial data. This fundamental problem in the mathematical theory of partial differential equations is known as the "Cauchy problem" after one of its first investigators. It stands in complete analogy to the general problem of a classical mechanical system whose evolution is determined by the initial positions and velocities of its elements. We wish to give in this chapter a detailed study of this initial-value problem, which will throw considerable light on the qualitative structure of the field equations.

8.1 Formulation of the Initial-Value Problem

Let us prescribe a three-dimensional hypersurface S oriented in space. We can then choose, without any loss of generality, a coordinate system such that the hypersurface S is described by the equation $x^0 = 0$. Since the normal to a surface oriented in space is itself oriented in time, our assumption forces g_{00} to be positive. Physically, we may interpret S as representing the space at the given time $x^0 = 0$. In this space we prescribe the components of the metric tensor $g_{\mu\nu}$ and their first derivatives. However, giving the values of the metric potentials all over S automatically allows us to compute all their derivatives taken in S (interior derivatives). That is, the derivatives $g_{\mu\nu|i}$, which do not contain any differentiation with respect to the index 0 (time), are known on S by differentiation of the given $g_{\mu\nu}$ in S. Therefore it is sufficient to give only the following initial values on S:

(8.3) $g_{\mu\nu}\,,\ g_{\mu\nu|0}$ (metric potentials and their normal derivatives)

These initial data, together with the differential system (8.1), form a typical Cauchy initial-value problem in the theory of partial differential equations, which we shall now discuss in detail.

8.2 Structure of Einstein's Equations

To investigate the Cauchy problem for Einstein's equations, we proceed in a standard manner always followed in initial-value problems for partial differential equations. We try to express the second time derivatives of our unknown functions in terms of the known space derivatives and the first-order time derivatives. If this can be done in analytic form, we may differentiate these identities indefinitely in time and obtain recursively all time and space derivatives of the $g_{\mu\nu}$ in S in terms of the given initial data. If the $g_{\mu\nu}$ admit a power-series development in x^0 near to S, we shall be able to compute all coefficients of this development by the above procedure. Thus this development will be uniquely determined by our initial data. Once one has achieved a continuation of the solution from an initial surface S into the future, one can try to study the evolution by indefinite continuation for all time and space.

Let us write out the contracted Riemann tensor in a form which displays clearly all the second derivatives of the metric tensor:

$$R_{\mu\nu} = \tfrac{1}{2}g^{\rho\sigma}[-g_{\mu\sigma|\nu|\rho} - g_{\nu\rho|\mu|\sigma} + g_{\mu\nu|\rho|\sigma} + g_{\rho\sigma|\mu|\nu}] + K_{\mu\nu} \tag{8.4}$$

Here the term $K_{\mu\nu}$ contains only the metric potentials and their first derivatives. It is due to the last two terms in (8.2) and to the remainder of terms in the first two Christoffel symbols which do not contain derivatives higher than first order. Hence $K_{\mu\nu}$ is known on S from the initial data. By differentiation in S, that is, in the space variables x^i, all higher derivatives of the $g_{\mu\nu}$ can be computed on S, as long as they do not involve more than one differentiation in time. All such derivatives are expressible by successive differentiation of the initial data on S, which must of course be assumed to be indefinitely differentiable.

We can next use Eqs. (8.1) and the form (8.4) of $R_{\mu\nu}$ to compute the second time derivatives of the metric potentials in terms of the known data on S. We observe that the indices $i, j = 1, 2, 3$ appear quite differently from the time index 0. An easy calculation shows

$$R_{ij} = \tfrac{1}{2}g^{00}g_{ij|0|0} + M_{ij} = 0 \tag{8.5a}$$

$$R_{i0} = -\tfrac{1}{2}g^{0j}g_{ij|0|0} + M_{i0} = 0 \tag{8.5b}$$

$$R_{00} = \tfrac{1}{2}g^{ij}g_{ij|0|0} + M_{00} = 0 \tag{8.5c}$$

where the $M_{\mu\nu}$ can be expressed in terms of the initial data on S.

We find the following surprising situation:

1. The linear system (8.5) for the calculation of the second time derivatives does not contain the unknowns $g_{\lambda 0|0|0}$ which are needed to determine the time evolution of the metric from the data on S. We have a problem of *underdetermination*.
2. The linear system (8.5) represents a set of 10 equations for the six unknowns $g_{ij|0|0}$ on S, which presents a problem of *overdetermination* and leads to compatibility requirements for the data $M_{\mu\nu}$ on S.

Let us start with problem 1. It is not surprising that the knowledge of $R_{\mu\nu} = 0$ does not determine the $g_{\mu\nu}$ in a unique way. Even in a flat space for which the entire Riemann tensor vanishes, the metric potentials $g_{\mu\nu}$ are still somewhat arbitrary, depending on our choice of coordinates. We shall now show that we can always make a coordinate transformation in the neighborhood of S such that the $g_{\mu\nu}$ and their first derivatives in S are unchanged but $g_{\lambda 0|0|0} \equiv 0$ on S. This fact explains why Eqs. (8.1) could not possibly contain information on these second time derivatives; indeed, the change of coordinates does not affect the tensor relations (8.1), which must be valid in all equivalent coordinate systems.

We introduce the coordinate transformation

$$\bar{x}^\lambda = x^\lambda + \tfrac{1}{6}(x^0)^3 A^\lambda(x^\alpha) \tag{8.6a}$$

where the $A^\lambda(x)$ represent four functions defined in the neighborhood of S which will be chosen to fit our requirements. Under the transformation (8.6*a*), the equation of S remains obviously $\bar{x}^0 = 0$, and we have on S

$$\frac{\partial \bar{x}^\lambda}{\partial x^\mu} = \delta^\lambda{}_\mu \qquad \left(\frac{\partial \bar{x}^\lambda}{\partial x^\mu}\right)_{|\nu} = 0 \qquad \text{on } S \tag{8.6b}$$

Even all second derivatives of the Jacobi matrix $\partial\bar{x}^\lambda/\partial x^\mu$ vanish on S except for

$$\left(\frac{\partial \bar{x}^\lambda}{\partial x^0}\right)_{|0|0} = A^\lambda(x^\alpha) \tag{8.6c}$$

We observe that the coordinate transformation (8.6*a*) has a nonvanishing Jacobian near S and is therefore an admissible transformation.

We now apply (8.6*b*) and (8.6*c*) to find the relation between the metric potentials and their first two derivatives in the two coordinate systems on the surface S. We have the identity

$$g_{\lambda\mu} = \bar{g}_{\alpha\beta}\,\frac{\partial \bar{x}^\alpha}{\partial x^\lambda}\,\frac{\partial \bar{x}^\beta}{\partial x^\mu} \tag{8.7}$$

and since the Jacobi matrix behaves on S like the unit matrix up to the second derivatives, we find *on S*

$$g_{\lambda\mu} = \bar{g}_{\lambda\mu} \qquad g_{\lambda\mu|\nu} = \bar{g}_{\lambda\mu|\nu} \qquad g_{\lambda\mu|\nu|\rho} = \bar{g}_{\lambda\mu|\nu|\rho} \tag{8.8}$$

except for the case $\nu = \rho = 0$. In this case, (8.6*b*) and (8.7) yield on S

$$g_{\lambda\mu|0|0} = \bar{g}_{\lambda\mu|0|0} + \bar{g}_{\alpha\mu}\left(\frac{\partial \bar{x}^\alpha}{\partial x^\lambda}\right)_{|0|0} + \bar{g}_{\lambda\beta}\left(\frac{\partial \bar{x}^\beta}{\partial x^\mu}\right)_{|0|0} \tag{8.9}$$

The last two terms in (8.9) vanish except for either λ or μ being zero. Using (8.6*c*), we thus arrive at the equations

$$g_{ij|0|0} = \bar{g}_{ij|0|0} \qquad g_{i0|0|0} = \bar{g}_{i0|0|0} + \bar{g}_{i\beta}A^\beta \qquad g_{00|0|0} = \bar{g}_{00|0|0} + 2\bar{g}_{0\beta}A^\beta \tag{8.10}$$

We choose now the as yet undetermined functions $A^\beta(x)$ in such a way that all $\bar{g}_{\lambda 0|0|0}$ vanish on S. For this purpose, we have to demand only that

$$g_{i0|0|0} = g_{i\beta}A^\beta \qquad \tfrac{1}{2}g_{00|0|0} = g_{0\beta}A^\beta \tag{8.11}$$

since we know that on S we have $g_{\alpha\beta} = \bar{g}_{\alpha\beta}$. The system (8.11) consists of four linear equations for the four unknowns A^β with the determinant $g = \det((g_{\mu\nu}))$ of the metric tensor itself. This determinant is nonzero, since we always assume that the metric is everywhere regular. Thus the $A^\beta(x)$ are determined by (8.11) at every point on S in a unique way.

Our purely formal analysis of the equation system (8.1) has led to an important insight into the causal meaning of these equations. The initial data on S do not determine the resulting metric in a unique way; the solution contains the four arbitrary functions $g_{\lambda 0|0|0}$ which are at our disposal. It should be observed that this arbitrariness is due to the fact that we can pick an arbitrary coordinate system for the description of the space-time continuum. However, the solutions obtained will differ only formally; they will describe the same geometrico-physical situation in different reference systems. This feature of the Einstein equations (8.1) was already stressed by Hilbert in 1915. He drew an important conclusion from this fact (Hilbert, 1915): Since the 10 differential equations (8.1) leave a freedom of four arbitrary functions in the solution, they cannot be entirely independent, but must have four inner relations. These relations are, of course, a consequence of the Bianchi identities for the full Riemann tensor and are known to us from Chap. 5 in the form of the condition that the Einstein tensor be divergenceless.

In the following we shall suppose that on the initial-value surface S we have chosen the metric tensor in such a way that $g_{\lambda 0|0|0} \equiv 0$. This normalization removes the first difficulty mentioned above of underdetermination in the field equations (8.1).

We return now to the system of differential equations (8.5), which contains 10 conditions for the six unknowns $g_{ij|0|0}$. We face here the problem of overdetermination. We observe that the first set of equations $R_{ij} = 0$ is sufficient to calculate the $g_{ij|0|0}$ and that the additional four equations $R_{i0} = 0$ and $R_{00} = 0$ must therefore be necessary consequences of this determination. To study this question in detail, we combine the first set of equations (8.5) with the second and third in the form

$$g^{00}R_{i0} + g^{0j}R_{ij} = g^{00}M_{i0} + g^{0j}M_{ij} = 0$$
$$g^{00}R_{00} - g^{ij}R_{ij} = g^{00}M_{00} - g^{ij}M_{ij} = 0 \qquad (8.12)$$

We see that these combinations, which must be zero by virtue of the field equations (8.1), depend only on the values of the components of the metric tensor and their first derivatives and upon the derivatives of these quantities with respect to the space variables. Hence Eqs. (8.12) represent four constraints on the initial data on S. As usual, if a system of equations does not determine a unique solution, one runs into a set of compatibility conditions for the data in order that a solution may be possible at all. Equations (8.12) are of this nature.

The structure of Eqs. (8.5) and (8.12) becomes very clear if we consider, besides the contracted Riemann tensor $R_{\alpha\beta}$, the Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$. We have, by definition and (8.12),

$$(8.13) \quad G_i{}^0 = R_i{}^0 = g^{0j}R_{ij} + g^{00}R_{i0} = g^{00}M_{i0} + g^{0j}M_{ij}$$

$$(8.14) \quad G_0{}^0 = R_0{}^0 - \tfrac{1}{2}R = g^{0\alpha}R_{0\alpha} - \tfrac{1}{2}(g^{ij}R_{ij} + g^{0i}R_{0i} + g^{i0}R_{i0} + g^{00}R_{00})$$

which simplifies to

$$(8.14') \quad G_0{}^0 = \tfrac{1}{2}(g^{00}R_{00} - g^{ij}R_{ij}) = \tfrac{1}{2}(g^{00}M_{00} - g^{ij}M_{ij})$$

Thus the equation system (8.5) may be written in the equivalent form

$$(8.15a) \qquad R_{ij} = 0$$

$$(8.15b) \qquad G_\lambda{}^0 = G^0{}_\lambda = 0$$

This normal form of the field equations is due to Lichnerowicz (1955). The set (8.15a) of six equations serves to determine the six unknown functions $g_{ij|0|0}$ from the initial data on S. The additional four equations (8.15b) in terms of the Einstein tensor $G_\lambda{}^0$ depend only on the initial data and represent necessary conditions on the initial data in order that a solution exist at all.

We have shown earlier that the differential equations for the metric field in empty space may be written in the two equivalent tensor forms

$R_{\mu\nu} = 0$ or $G_{\mu\nu} = 0$. The new system (8.15) is not covariant and is formally inferior to the other two formulations. However, in a fixed coordinate system it is particularly convenient for the study of the initial-value problem, as we shall see in the next section.

8.3 Separation of the Cauchy Problem into Two Parts

So far, we have given simple formulas for all second derivatives of the metric tensor on S as follows:

$g_{ij\|0\|0} = \dfrac{-2M_{ij}}{g^{00}}$	from (8.5) since $g^{00} > 0$
$g_{\lambda 0\|0\|0} = 0$	by normalization of solution
$g_{\alpha\beta\|i\|\lambda}$	known by interior differentiation of initial data on S

On the other hand, we found that the initial data on S must satisfy the compatibility condition $G^0{}_\alpha = 0$.

If we wish to extend the solution into space-time outside of the hypersurface S, we have to use the equation system (8.5), or equivalently the two sets of equations (8.15a) and (8.15b). We shall now prove a remarkable fact, namely, that once equations (8.15b) are satisfied on the initial hypersurface S, they will remain automatically valid for all time. Thus Eqs. (8.15b) are essentially a consequence of the more fundamental system (8.15a). We may interpret $G^0{}_\alpha = 0$ as integrals of the differential system (8.15a).

To prove our assertion, we start with the Einstein tensor, which is by definition $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}$, and use the fact that, by virtue of (8.15a), all R_{ij} vanish. Hence we easily find

$$G^i{}_j = g^{i0}R_{0j} - \tfrac{1}{2}g^i{}_j(g^{00}R_{00} + 2g^{0l}R_{0l})$$
$$G^i{}_0 = g^{i0}R_{00} + g^{il}R_{l0} \qquad (8.16)$$

and

$$(8.17) \qquad G^0{}_j = g^{00}R_{0j} \qquad G^0{}_0 = \tfrac{1}{2}g^{00}R_{00}$$

Observe that the right-hand sides in (8.16) depend only on $R_{0\lambda}$ and can be expressed by means of (8.17) in terms of $G^0{}_\lambda$. We see that $G^i{}_\lambda$ depends linearly upon $G^0{}_\lambda$, with coefficients which are easily expressed in terms of the metric tensor.